COMPARISON BETWEEN A CLASSICAL AND A BAYESIAN APPROACH FOR ESTIMATING HARMONICS OF PERIODIC ARBITRARY SIGNALS

G. A. Kyriazis [1], *P. M. Ramos* [2], *A. C. Serra* [3]

[1] Instituto Nacional de Metrologia, Normalização e Qualidade Industrial (Inmetro), Duque de Caxias, Brazil, gakyriazis@inmetro.gov.br
[2] Instituto de Telecomunicações, Instituto Superior Técnico, Universidade Técnica de Lisboa, Lisboa, Portugal, pedro.m.ramos@ist.utl.pt
[3] Instituto de Telecomunicações, Instituto Superior Técnico, Universidade Técnica de Lisboa, Lisboa, Portugal, acserra@ist.utl.pt

Abstract: Two methods for the estimation of harmonic parameters of periodic arbitrary signals were compared: a method based on least squares and a Bayesian method using approximations based on the normal distribution and the posterior mode. The methods are reviewed and comparison results are presented and analyzed in this paper.

Keywords: harmonics, sampling methods, Bayesian inference, signal analysis, least squares.

1. INTRODUCTION

An algorithm based on least squares for fitting the parameters of arbitrary signals described as a sum of sinusoids with harmonically related frequencies (or with non-harmonically related frequencies but whose ratios are known) to sampled data was presented in [1]. It is an extension of the four-parameter sine-fitting algorithm [2]. The convergence limitations were studied and the algorithm was further improved in [3]. A similar algorithm had been applied independently to the calibration of thermal converters [4] and of electric power and energy meters [5][6].

Bayesian parameter estimation techniques have been applied to spectrum analysis [7] and, only more recently, their useful application to waveform metrology has been pointed out [8]. The method is applicable to any complex trigonometric signal: sum of sinusoids with harmonically related frequencies (or with non-harmonically related frequencies whose ratios are known or not), multi-tone amplitude or frequency modulation, fluctuating harmonics or even chirped signals. The method is applied here to the specific problem of estimating the harmonics of periodic arbitrary signals.

This problem of inference has many applications in the calibration or performance evaluation of data converters, waveform recorders, signal generators and analyzers, sources of phantom electrical power distorted signals, and power quality measuring instruments. It has also been applied to impedance measurements and to impedance spectroscopy.

The algorithm described in [3] was sent by Instituto Superior Técnico from Universidade Técnica de Lisboa (IST - UTL) to Instituto Nacional de Metrologia, Normalização e Qualidade Industrial (Inmetro) as an executable software module developed in LabVIEW. The objective was to compare the classical and Bayesian techniques for estimating signal harmonics from computer simulated sampling data. Both techniques are reviewed in section 2 and comparison results are presented and analyzed in section 3. The conclusions are drawn in section 4.

2. ESTIMATION

A total of N uniform samples are taken at times $i = 0, \ldots, N-1$ (dimensionless units). The time series is postulated to contain a signal $f[i]$ with additive noise $e[i]$. It is assumed that the data $y[i]$ can be modeled as

$$y[i] = f[i] + e[i]$$
$$f[i] = C_0 + \sum_{j=1}^{m} \left(A_j \cos[j\omega i] + B_j \sin[j\omega i]\right) \tag{1}$$

where C_0 is the dc component, A_j and B_j are the j-th harmonic amplitudes, $j\omega$ is the j-th harmonic of the fundamental frequency ω, and m is the number of harmonics. The total number of functions $(2m+1)$ is assumed known.

The estimates of the harmonic amplitudes are here reported in polar coordinates, that is

$$a_j = \left[\left(A_j^2 + B_j^2\right)\right]^{1/2} \tag{2a}$$
$$\theta_j = \tan^{-1}\left[-B_j / A_j\right] \tag{2b}$$

where a_j and θ_j are respectively the j-th harmonic amplitude and phase angle ($j = 0, \ldots, m$). The first element, a_0, is the dc component C_0 and therefore $\theta_0 = 0$.

Model (1) can be written in matrix notation as

$$\mathbf{y} = \mathbf{D}(\omega)\mathbf{x} + \mathbf{e}, \tag{3a}$$

where matrix $\mathbf{D}(\omega)$ has N rows and $2m+1$ columns, that is

$$\mathbf{D}(\omega) = [\mathbf{1} \quad \mathbf{c}_1 \quad \mathbf{s}_1 \quad \cdots \quad \mathbf{c}_m \quad \mathbf{s}_m], \tag{3b}$$

$\mathbf{y} = (y[0], \ldots, y[N-1])'$ is the data vector, $\mathbf{c}_j = \cos[j\omega\mathbf{t}]$, $\mathbf{s}_j = \sin[j\omega\mathbf{t}]$, $\mathbf{t} = (0, \ldots, N-1)'$ is the vector of time indexes, $\mathbf{1}$ is a vector with all elements equal to one, $\mathbf{x}$ is the $(2m+1)$-vector of fitting parameters, i.e. $\mathbf{x} = (C_0, A_1, B_1, A_2, B_2, \ldots, A_m, B_m)'$, and $\mathbf{e} = (e[0], \ldots, e[N-1])'$ is the vector of noise terms. The prime (´) means transpose. It is assumed that ω and $\mathbf{x}$ are unknown.

It is also assumed that the elements of **e** are normally and independently distributed, each with mean zero and unknown common variance σ^2; that is, $E[\mathbf{e}] = 0$ and $E[\mathbf{ee}'] = \sigma^2\mathbf{I}_N$, where $\mathbf{I}_N$ is the identity matrix of order N.

2.1 Bayesian analysis

Since $\mathbf{D}'(\omega)\mathbf{D}(\omega)$ is symmetric, and assuming positive definiteness, it can be orthogonalized, so that

$$\mathbf{W}(\omega) = \mathbf{D}(\omega)\mathbf{Q}(\omega)\mathbf{\Lambda}^{-1/2}(\omega) \tag{4}$$

is a rectangular matrix with orthonormal columns, where $\mathbf{Q}(\omega)$ and $\mathbf{\Lambda}(\omega)$ are, respectively, the eigenvector and eigenvalue matrices. Model (3) may be rewritten in terms of $\mathbf{W}(\omega)$ and the **z** parameters of the orthogonal model as

$$\mathbf{y} = \mathbf{W}(\omega)\mathbf{z} + \mathbf{e} \tag{5}$$

Assuming noninformative priors for the parameters (ω, **z**, σ), the posterior marginal probability density function (pdf) for ω is [7]

$$p(\omega \mid \mathbf{y}) \propto \left[1 - \frac{\|\mathbf{W}'(\omega)\mathbf{y}\|^2}{\|\mathbf{y}\|^2}\right]^{(n-N)/2} \tag{6}$$

where the symbol $\|\cdot\|$ denotes the length of a vector. This Student t pdf expresses all the information about the frequency that is contained in the prior information and data. The function $\|\mathbf{W}'(\omega_0)\mathbf{y}\|^2$ plays the role of a ´sufficient statistic´; it reduces to the periodogram when the data contains a sinusoidal signal.

The Student t pdf (6) can be approximated by a normal pdf with mean $\hat{\omega}$ and variance $\sigma_\omega^2 = \sigma^2/b$, where

$$b = -\frac{1}{2}\frac{d^2\|\mathbf{W}'(\omega)\mathbf{y}\|^2}{d\omega^2}\Bigg|_{\omega=\hat{\omega}} \tag{7}$$

As the log of the Student t pdf is so sharply peaked when the sample size is large and/or the signal-to-noise ratio (SNR) is high, use is made of an improved version of the Hooke and Jeeves´ pattern search algorithm [9] to find its mode $\hat{\omega}$.

The user is asked to input an estimate of the signal frequency in radians per sample and this estimate is used as a ´starting guess´ $\omega^{(0)}$. This can easily be done in our simulation examples as the true signal frequency is known. In metrology applications, the frequency setting of the digital generator is typically used as the starting point. Algorithm convergence is ensured and aliasing errors are avoided when the sampling theorem is satisfied, i.e., when $m\omega < \pi$. Moreover, the number of samples should be such that they cover more than two signal periods, i.e., $\omega N > 4\pi$. The reduction factor ρ for step size and the "minimum" step size ε of the pattern search algorithm are also set to ensure algorithm convergence (see [9]).

At each iteration n, matrix $\mathbf{D}'(\omega^{(n)})\mathbf{D}(\omega^{(n)})$ is orthogonalized by evaluating $\mathbf{Q}(\omega^{(n)})$ and $\mathbf{\Lambda}(\omega^{(n)})$, matrix $\mathbf{W}(\omega^{(n)})$ is calculated from (4) and the value of the objective function, i.e. the negative logarithm of (6), is computed. The pattern search algorithm is used to find the estimate $\hat{\omega}$ that minimizes this function. The algorithm works by taking "steps" from one estimate $\omega^{(n-1)}$ of a minimum to another (hopefully better) estimate $\omega^{(n)}$. It assesses the frequency corrections in each iteration, and iterations are stopped when the absolute relative frequency correction is below the "minimum" step size predefined. The noise variance is estimated as $\hat{\sigma}^2 = \nu s^2(\hat{\omega})/(\nu - 2)\ (1 \pm u)$, where $u = [2/(\nu - 4)]^{1/2}$, $\nu = N - 2m - 1$ and $\nu s^2(\hat{\omega}) = \|\mathbf{y}\|^2 - \|\mathbf{W}'(\hat{\omega})\mathbf{y}\|^2$. The variance associated with the estimate $\hat{\omega}$ is then evaluated from (7) by replacing σ^2 with its estimate $\hat{\sigma}^2$.

The $(2m+1)$-vector of fitting parameters, i.e. $\mathbf{x} = (C_0, A_1, B_1, A_2, B_2, \ldots, A_m, B_m)'$, is estimated as $\hat{\mathbf{x}} = \mathbf{Q}(\hat{\omega})\mathbf{\Lambda}^{-1/2}(\hat{\omega})\mathbf{W}'(\hat{\omega})\mathbf{y}$. The estimates of the j-th amplitude $\hat{a}_j$ and phase angle $\hat{\theta}_j$ are computed from (2). The covariance matrix associated with $\hat{\mathbf{x}}$ is evaluated from $\mathbf{S_X} = \hat{\sigma}^2\mathbf{Q}(\hat{\omega})\mathbf{\Lambda}^{-1}(\hat{\omega})\mathbf{Q}'(\hat{\omega})$. We confirmed that $\mathbf{S_X}$ is nearly diagonal for large SNR. The uncertainties of the j-th amplitudes and phase angles can then be easily evaluated from (2) using the GUM approximation rules [10], namely, $[(\mathbf{S_X})_{jj}]^{1/2}$ for the j-th amplitude and $|\hat{a}_j|^{-1}[(\mathbf{S_X})_{jj}]^{1/2}$ for the j-th phase angle, where the symbol $(\cdot)_{jj}$ denotes the j-th diagonal entry.

2.2 Least squares

Least-squares algorithms minimize the squared error between the sampled data and the fitting model. Two different least-squares algorithms for the estimation of sinusoidal signal parameters from sampled data were standardized in [2]. The first algorithm (commonly called the three-parameter sine-fitting algorithm) estimates the in-phase (A) and in-quadrature (B) amplitudes and the dc component when the frequency is known. It is a simple multiple linear regression since it is possible to linearly separate the contributions of the three parameters in the fitting model. Its main drawback is that the frequency must be known. In many acquisition systems, the frequency is not known either because the signal source is unknown, because the signal frequency cannot be measured with the necessary accuracy or even because the actual sampling rate is not accurately known (this is an issue because all acquisition-based algorithms for estimating frequency actually estimate the ratio between the signal frequency in radians per sample and the sampling rate in samples per second).

The second algorithm addresses these issues by also estimating the frequency. It is commonly called the four-parameter sine-fitting algorithm because it estimates four parameters (two amplitudes, one dc component and one frequency). However, since the frequency is included as a parameter to be estimated, it is no longer possible to linearly separate completely the four parameters in the fitting model. This is solved through a first order Taylor series expansion so that the algorithm estimates a frequency correction in each iteration.

An extension of these algorithms to multi-harmonic signals was presented in [1]. Much like the single-tone algorithms, there are also two versions. In the first one (typically called the $2m+1$ algorithm) the frequency must be supplied to the algorithm and it estimates the amplitudes and phases of m harmonics and the dc component). It is necessary to indicate to the algorithm which harmonics (or the ratios between the frequency components) are to be estimated. The harmonics estimated can be only a few ones selected using prior knowledge of the signal harmonic composition or the algorithm can estimate a range of harmonics (even if their amplitudes in the signal are negligible). This algorithm is non-iterative as it corresponds to a multiple linear regression.

The other version that also estimates the frequency is typically called the $2m$+2 algorithm. As with the single-tone algorithm it is iterative and adjusts the frequency value in each iteration.

The criteria used to stop the iterations for the single-tone algorithm in [2] is based on the evaluation of the total root-mean-square (RMS) error between the sampled data and the fitting model. When this error falls below a predefined threshold the iterations are stopped. However, the definition of this threshold is quite complex since it depends on the actual noise in the signal. Defining a very low threshold below the noise level means that the algorithm will not stop. If the threshold is defined too high, the algorithm may stop too early while it still could improve the estimates.

A different method was proposed in the algorithms for multi-harmonic signals to avoid these problems. In the same way as described in the previous section, the algorithm assesses the frequency corrections in each iteration, and iterations are stopped when the absolute relative frequency correction is below a threshold. This threshold is easier to define since it depends on the desired frequency correction (this might also be dependent on the noise but it is easier to set). Since the error is no longer used to stop the algorithm, there is no need to reconstruct the signal in each iteration. This difference can be very useful for implementations in stand-alone instruments.

The fitting model (1) includes m in-phase amplitudes A_j, m in-quadrature amplitudes B_j, one dc component C_0 and one signal frequency ω. The FFT and then the IpDFT [11] are used to estimate the signal frequency from the data vector $\mathbf{y}$ and this estimate is used as an initial value. This value is then used in the $2m$+1 algorithm to estimate the initial values of the m harmonic amplitudes. At each iteration n of the $2m$+2 algorithm, a matrix $\mathbf{M}^{(n)}$ is built and used to estimate the fitting parameters from

$$\mathbf{x}^{(n)} = \left(\mathbf{M}'^{(n)}\mathbf{M}^{(n)}\right)^{-1}\mathbf{M}'^{(n)}\mathbf{y}\ , \tag{8}$$

where $\mathbf{x}^{(n)}$ is the ($2m$+2)-vector of fitting parameters at the n-th iteration, i.e. $\mathbf{x}^{(n)} = (C_0^{(n)}, A_1^{(n)}, B_1^{(n)}, A_2^{(n)}, B_2^{(n)},\ldots, A_m^{(n)}, B_m^{(n)}, \Delta\omega^{(n)})'$, and $\Delta\omega^{(n)}$ is the frequency correction at the n-th iteration. The latter is used to update the frequency estimate of the previous iteration,

$$\omega^{(n)} = \omega^{(n-1)} + \rho\Delta\omega^{(n)}, \tag{9}$$

where ρ is the reduction factor for the frequency correction [3]. Matrix $\mathbf{M}^{(n)}$ has N rows and $2m$+2 columns, that is

$$\mathbf{M}^{(n)} = \left[\mathbf{D}\left(\omega^{(n)}\right) \ \vdots\ \sum_{j=1}^{m}\boldsymbol{\alpha}_j\right] \tag{10}$$

where $\boldsymbol{\alpha}_j = -A_j^{(n-1)}j(\mathbf{t}\circ\mathbf{s}_j) + B_j^{(n-1)}j(\mathbf{t}\circ\mathbf{c}_j)$, and ($\circ$) denotes the Hadamard product (or entrywise product).

The iterative process ends when the absolute relative frequency correction is below a predefined threshold

$$\left|\frac{\Delta\omega^{(n)}}{\omega^{(n)}}\right| < \varepsilon\ . \tag{11}$$

3. COMPARISON RESULTS

A comparison was made between the Bayesian method [8] and the classical one based on least squares [3]. The former was implemented in LabWindows/CVI v. 6.0 and the latter in LabVIEW 2009. The times reported refer to the 3 GHz, 1.96 GB random-access memory (RAM), Duo Core computer used to process the data under a Windows XP® environment. The objective was to estimate the harmonic parameters of periodic arbitrary signals. Two sets of samples have been selected for this purpose.

The first set of samples was generated from

$$y[i] = 0.001 + \cos\,[\omega i + 1] + 0.1\sum_{j=2}^{50}\sin\,[j\omega i + 3e[j]] + e[i]$$

where $e[j] \sim N(0, 1)$ is the j-th (pseudo) random phase angle. The uncertainties associated with the estimates decrease with increasing SNR and/or number of samples. Here, $N = 512$ and $e[i] \sim N(0,\ 1\cdot10^{-6})$. The latter is the order of magnitude of the noise variance typically found in calibration systems that employ commercial signal generators. See Table I (or Table III) for the true value of the signal frequency. The signal waveform corresponds to (1) with m = 50. The sampling theorem is satisfied ($50\cdot\omega < \pi$) and the number of samples covers more than two signal periods ($512\cdot\omega > 4\pi$).

The second set of $N = 512$ samples was generated from NPL worst case voltage waveform [12] with additive noise $e[i] \sim N(0,\ 1\cdot10^{-6})$. See Table II (or Table IV) for true values of the signal frequency and harmonic amplitudes. The signal waveform corresponds to (1) with $m = 49$ and with no even harmonics except for the second and the fourth. Again the sampling theorem is satisfied and the number of samples covers more than two signal periods.

3.1 Bayesian analysis

For $m = 50$ there are 100 amplitudes, one dc component, one frequency and one noise variance. Probability theory has eliminated all parameters except the fundamental frequency. The pattern search algorithm was used to locate the maximum of the one-dimensional posterior pdf for ω. The algorithm converged to the desired maximum with any starting point within $0.0623 < \omega^{(0)} < 0.0633$. The larger the number of harmonics is, the greater the sensitivity of the optimization algorithm to starting point changes. The reduction factor ρ and the "minimum" step size ε were set at 0.50 and $1\cdot10^{-8}$, respectively. The processing time was 5 s.

Tables I and II show the ´best´ estimates for the first and second set of samples, respectively. For the first set, the estimates are reported only for the first 10 harmonic amplitudes due to space limitations. The estimates are similar for the remaining harmonics. The uncertainty assessment is an automatic outcome of the Bayesian framework. The data and the corresponding residuals are presented in Figs. 1 and 2.

Table I - Estimates obtained from Bayesian analysis (first set).

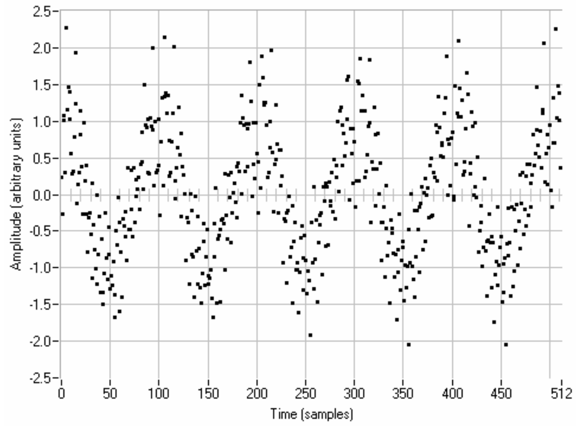
Parameter	True values	Estimate	Uncertainty
ω	0.062812345	0.062812346	0.000000022
C_0	0.001	0.000983	0.000047
a_1	1	0.999995	0.000094
θ_1	1	1.000081	0.000094
a_2	0.1	0.100009	0.000094
a_3	0.1	0.100052	0.000094
a_4	0.1	0.099966	0.000094
a_5	0.1	0.099909	0.000094
a_6	0.1	0.099943	0.000094
a_7	0.1	0.099969	0.000094
a_8	0.1	0.099951	0.000094
a_9	0.1	0.100070	0.000094
a_{10}	0.1	0.099969	0.000094
σ	0.001	0.001061	0.000074

Note: frequency unit - radian per sample; amplitude unit - arbitrary unit; phase unit: radian; noise standard deviation unit – same unit as that used for the amplitudes.

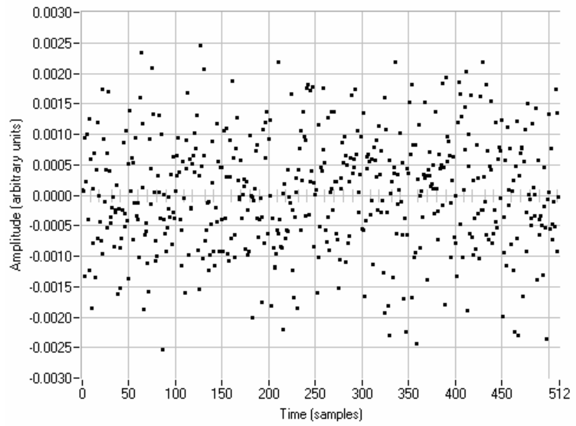
Table II - Estimates obtained from Bayesian analysis (second set).

Parameter	True values	Estimate	Uncertainty
ω	0.062812345	0.062812354	0.000000047
C_0	0	0.000024	0.000044
a_1	1	0.999959	0.000094
θ_1	0	0.000000	0.000094
a_2	0.49984	0.499883	0.000094
a_3	0.50000	0.500071	0.000094
a_4	0.24976	0.249805	0.000094
a_5	0.30000	0.299910	0.000094
a_7	0.20000	0.199995	0.000094
a_9	0.20000	0.200077	0.000094
a_{11}	0.10000	0.099996	0.000094
a_{13}	0.10000	0.099936	0.000094
a_{15}	0.10000	0.099929	0.000094
a_{17}	0.10000	0.099893	0.000094
a_{19}	0.10000	0.100027	0.000094
a_{21}	0.09000	0.090008	0.000094
a_{23}	0.08000	0.079991	0.000094
a_{25}	0.07000	0.069901	0.000094
a_{27}	0.07000	0.070118	0.000094
a_{29}	0.06000	0.059945	0.000094
a_{31}	0.06000	0.060002	0.000094
a_{33}	0.03465	0.034607	0.000094
a_{35}	0.05000	0.050006	0.000094
a_{37}	0.05000	0.049939	0.000094
a_{39}	0.05000	0.049980	0.000094
a_{41}	0.05000	0.049976	0.000094
a_{43}	0.04000	0.040041	0.000094
a_{45}	0.04000	0.040009	0.000094
a_{47}	0.04000	0.039976	0.000094
a_{49}	0.04000	0.040006	0.000094
σ	0.001	0.001060	0.000074

Note: frequency unit - radian per sample; amplitude unit - arbitrary unit; phase unit: radian; noise standard deviation unit – same unit as that used for the amplitudes.

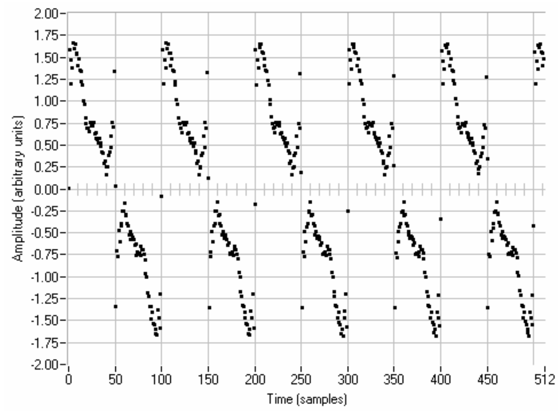


(a)

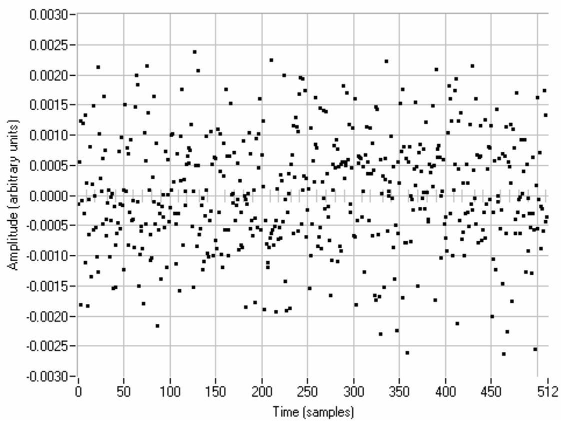


(b)

Fig. 1. First set of samples (m = 50). (a) Data. (b) Residuals.



(a)



(b)

Fig. 2. Second set of samples (m = 49). (a) Data. (b) Residuals.

3.2 Least squares

The reduction factor ρ and the threshold ε were set at 0.10 and $1{\cdot}10^{-9}$, respectively. The least squares procedure ran much faster. Its processing time was much less than 1 s. Tables III and IV list the estimates as displayed by the software (no attempt was made to reduce them to significant figures). The reader should compare the results with those listed in Tables I and II. Bayesian estimates indeed do not differ significantly from the least squares estimates.

The principle of least squares provides no way to assess the uncertainty associated with the estimates other than the sampling distribution of the estimator, which refers to an imaginary class of different data sets and not to the specific one at hand. Bounds for the variance of the estimators can be derived when the noise variance is known. The Cramér-Rao lower bounds (CRLB) listed in Tables III and IV were evaluated from [13] and [14] by replacing σ^2 with the estimate $\hat{\sigma}^2$ obtained in the previous section.

Moreover, the least squares algorithms described and/or adopted in [1]-[6] are restricted to the estimation of sinusoidal signals or arbitrary signals described as a sum of sinusoids whose frequencies have known ratios. The Bayesian approach in [7] and [8] is much more versatile. As shown in [8], essentially the same algorithm can be applied to any complex trigonometric signal the user may conceive.

4. CONCLUSIONS

As there is little prior information beyond the signal waveform, Bayesian estimates of signal parameters do not differ significantly from the estimates one obtains from least squares. The uncertainty evaluation is an automatic outcome of the Bayesian framework. In contrast, the principle of least squares provides no way to assess the uncertainty associated with the estimates other than the sampling distribution of the estimator, which refers to imaginary data sets and not to the specific one at hand.

The least squares algorithm employed here is much simpler and faster than the Bayesian one. However, it is restricted to the estimation of sinusoidal signals or arbitrary signals described as a sum of sinusoids whose frequencies have known ratios. The Bayesian approach applied here is much more versatile. Essentially the same algorithm can be applied to any complex trigonometric signal the user may conceive.

Table III - Estimates obtained from least squares (first set).

Parameter	True values	Estimate	CLRB
ω	0.062812345	0.062812	$2.0{\cdot}10^{-13}$
C_0	0.001	0.00098	$2.2{\cdot}10^{-9}$
a_1	1	0.999996	$4.4{\cdot}10^{-9}$
θ_1	1	1.00007	$1.7{\cdot}10^{-8}$
a_2	0.1	0.100000	$4.4{\cdot}10^{-9}$
a_3	0.1	0.100052	$4.4{\cdot}10^{-9}$
a_4	0.1	0.099967	$4.4{\cdot}10^{-9}$
a_5	0.1	0.0999094	$4.4{\cdot}10^{-9}$
a_6	0.1	0.0999426	$4.4{\cdot}10^{-9}$
a_7	0.1	0.0999695	$4.4{\cdot}10^{-9}$
a_8	0.1	0.0999511	$4.4{\cdot}10^{-9}$
a_9	0.1	0.100069	$4.4{\cdot}10^{-9}$
a_{10}	0.1	0.099969	$4.4{\cdot}10^{-9}$
σ^2	0.000001	0.00000112	-

Table IV - Estimates obtained from least squares (second set).

Parameter	True values	Estimate	CLRB
ω	0.062812345	0.062812	$2.0{\cdot}10^{-13}$
C_0	0	0.000018	$2.2{\cdot}10^{-9}$
a_1	1	0.999959	$4.4{\cdot}10^{-9}$
θ_1	0	0.000000	$1.7{\cdot}10^{-8}$
a_2	0.49984	0.499883	$4.4{\cdot}10^{-9}$
a_3	0.50000	0.500071	$4.4{\cdot}10^{-9}$
a_4	0.24976	0.249805	$4.4{\cdot}10^{-9}$
a_5	0.30000	0.299911	$4.4{\cdot}10^{-9}$
a_7	0.20000	0.199995	$4.4{\cdot}10^{-9}$
a_9	0.20000	0.200077	$4.4{\cdot}10^{-9}$
a_{11}	0.10000	0.0999964	$4.4{\cdot}10^{-9}$
a_{13}	0.10000	0.0999366	$4.4{\cdot}10^{-9}$
a_{15}	0.10000	0.0999292	$4.4{\cdot}10^{-9}$
a_{17}	0.10000	0.0998925	$4.4{\cdot}10^{-9}$
a_{19}	0.10000	0.100027	$4.4{\cdot}10^{-9}$
a_{21}	0.09000	0.0900078	$4.4{\cdot}10^{-9}$
a_{23}	0.08000	0.079991	$4.4{\cdot}10^{-9}$
a_{25}	0.07000	0.0699006	$4.4{\cdot}10^{-9}$
a_{27}	0.07000	0.0701178	$4.4{\cdot}10^{-9}$
a_{29}	0.06000	0.059945	$4.4{\cdot}10^{-9}$
a_{31}	0.06000	0.0600018	$4.4{\cdot}10^{-9}$
a_{33}	0.03465	0.0346068	$4.4{\cdot}10^{-9}$
a_{35}	0.05000	0.0500061	$4.4{\cdot}10^{-9}$
a_{37}	0.05000	0.0499385	$4.4{\cdot}10^{-9}$
a_{39}	0.05000	0.04998	$4.4{\cdot}10^{-9}$
a_{41}	0.05000	0.0499759	$4.4{\cdot}10^{-9}$
a_{43}	0.04000	0.0400413	$4.4{\cdot}10^{-9}$
a_{45}	0.04000	0.0400086	$4.4{\cdot}10^{-9}$
a_{47}	0.04000	0.039976	$4.4{\cdot}10^{-9}$
a_{49}	0.04000	0.0400059	$4.4{\cdot}10^{-9}$
σ^2	0.000001	0.00000112	-

REFERENCES

[1] P. M. Ramos, M. F. da Silva, R. C. Martins and A. Cruz Serra, “Simulation and experimental results of multiharmonic least-squares fitting algorithms applied to periodic signals,” *IEEE Trans. Instrum. Meas.*, vol. 55, no. 2, pp. 646-51, Apr. 2006.

[2] *Standard for Digitizing Waveform Records,* IEEE Std. 1057-1994.

[3] P. M. Ramos and A. Cruz Serra, “Least squares multiharmonic fitting: convergence improvements,” *IEEE Trans. Instrum. Meas.*, vol. 56, no. 4, pp. 1412-8, Aug. 2007.

[4] U. Pogliano, “Precision measurement of ac voltage below 20 Hz at IEN,” *IEEE Trans. Instrum. Meas.*, vol. 46, no. 2, pp. 369-372, Apr. 1997.

[5] U. Pogliano, “Use of integrative analog-to-digital converters for high-precision measurement of electrical power,” *IEEE Trans. Instrum. Meas.*, vol. 50, no. 5, pp. 1315-8, Oct. 2001.

[6] L. Di Lillo, H. Laiz, E. Yasuda and R. García, “Sampling wattmeter at INTI,” In *Proc. 8th International Seminar on Electrical Metrology (VIII Semetro)*, João Pessoa, Brazil, June 17-19, 2009

[7] G. L. Bretthorst, “Bayesian Spectrum Analysis and Parameter Estimation,” in *Lecture Notes in Statistics*, vol. 48, J. Berger et al, ed. New York: Springer-Verlag, 1988.

[8] G. A. Kyriazis, “A simple algorithm for Bayesian signal analysis with applications to metrology,” *IEEE Trans. Instrum. Meas.*, vol. 60, no. 7, pp. 2314-19, July 2011.

[9] M. G. Johnson, *Nonlinear optimization using the algorithm of Hooke and Jeeves*, 1994, accessed Jan. 29, 2010. [Online]. Available: http://www.netlib.org/opt/hooke.c/

[10] BIPM, IEC, IFCC, ISO, IUPAC, IUPAP and OIML 1995 *Guide to the Expression of Uncertainty in Measurement* (Geneva, Switzerland: International Organization for Standardization)

[11] J. Schoukens, R. Pintelon and H. Van Hamme, “The interpolated fast Fourier transform: A comparative study,” *IEEE Trans. Instrum. Meas.*, vol. 41, no. 2, pp. 226–232, Apr. 1992.

[12] National Physical Laboratory, *NPL Power Quality Waveform Library*, assessed Jul. 5, 2011. [Online]. Available: http://resource.npl.co.uk/ waveform/

[13] T. Andersson and P. Händel, "IEEE Standard 1057, Cramér-Rao bound and parsimony principle," *IEEE Trans. Instrum. Meas.*, vol. 55, no. 1, pp. 44-53, Feb. 2006.

[14] P. Stoica, R. L. Moses, B. Friedlander and T. Soderstrom, "Maximum likelihood estimation of the parameters of multiple sinusoids from noisy measurements," *IEEE Trans. Acoust., Speech, Signal Processing*, vol. 37, no. 3, pp. 378-92, Mar. 1989.