

DIRECT AND INDIRECT ESTIMATION OF A SINE WAVE AMPLITUDE BY THE ENERGY-BASED METHOD

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Abstract: The Energy-Based Method (EBM) provides both direct and indirect estimates of the amplitude of a sine wave. In this paper the accuracies of these two different estimation procedures are compared when some of the most commonly used cosine class windows are adopted. The comparison is performed by taking into account the effects of the algorithm error, spectral interference, and wideband noise. Based on the comparison results some conclusions of interest in the algorithm design phase are drawn. Such conclusions are finally validated by experimental results.

Key words: amplitude estimation, cosine windows, energy-based method, noncoherent sampling.

1. INTRODUCTION

Very accurate estimation of sine wave parameters are often required in many engineering applications. For this purpose, frequency-domain methods are usually preferred since they are robust with respect to signal model uncertainty and easily to implement. Among these procedures, the so-called Energy-Based Method (EBM) [1] is widely used because it is simple to understand and to apply. Moreover it is very fast and accurate. Using the EBM, the amplitude of a sine wave can be estimated both directly from selected spectral samples or indirectly by estimating at first the sine wave normalized frequency. Unfortunately, only the performance of the direct estimation procedure has been analyzed in the scientific literature [1–5]. However, both amplitude estimation procedures are affected by the algorithm error, the spectral interference from the image component, and wideband noise [2], [3], [5], [6]. The aim of this paper is to analyze and compare the effects of these uncertainty contributions on the accuracy of the provided estimates when some of the most commonly used cosine class windows are considered.

2. AMPLITUDE ESTIMATION BY THE EBM

Let us consider a sine wave of amplitude A , frequency f_{in} , and phase φ , sampled at frequency f_s , i.e.

$$x(m) = A \sin\left(2\pi \frac{f_{in}}{f_s} m + \varphi\right), \quad m = 0, 1, 2, \dots \quad (1)$$

The frequency f_{in} is chosen smaller than $f_s/2$ to satisfy the Nyquist theorem. When M samples are acquired, the ratio between the frequencies f_{in} and f_s can be expressed as:

$$\frac{f_{in}}{f_s} = \frac{\lambda_0}{M} = \frac{l + \delta}{M}, \quad (2)$$

where l and δ ($-0.5 \leq \delta < 0.5$) are respectively the integer and the fractional parts of the number of acquired sine wave cycles λ_0 . It is well-known that $\delta = 0$ corresponds to the so called coherent sampling. When $\delta \neq 0$, as occurs in most practical applications, the spectrum of the signal $x(\cdot)$ is affected by spectral leakage [1–3].

To reduce spectral leakage errors, windowing is often employed. Thus, the analyzed signal becomes $x_w(m) = x(m) \cdot w(m)$, $m = 0, 1, 2, \dots, M$, where $w(\cdot)$ is the adopted window, usually belonging to the cosine class [1], [7]. The H -term cosine window is defined as:

$$w(m) = \sum_{h=0}^{H-1} (-1)^h a_h \cos\left(2\pi h \frac{m}{M}\right), \quad m = 0, 1, \dots, M-1 \quad (3)$$

where a_h , $h = 0, \dots, H-1$ are the window coefficients.

The Discrete-Time Fourier Transform (DTFT) of the signal $x_w(\cdot)$ is given by [1–3]:

$$X_w(\lambda) = \frac{A}{2j} \left[W(\lambda - \lambda_0) e^{j\varphi} - W(\lambda + \lambda_0) e^{-j\varphi} \right], \quad \lambda \in [0, M] \quad (4)$$

where λ is the continuous frequency expressed in bins and $W(\cdot)$ is the DTFT of the window $w(\cdot)$.

For the H -term cosine windows we have [8]:

$$W(\lambda) = \sin(\pi\lambda) e^{-j\pi \frac{M-1}{M} \lambda} \times \sum_{h=0}^{H-1} (-1)^h 0.5 a_h \left[\frac{e^{-j\pi \frac{h}{M} \lambda}}{\sin \frac{\pi}{M} (\lambda - h)} + \frac{e^{j\pi \frac{h}{M} \lambda}}{\sin \frac{\pi}{M} (\lambda + h)} \right], \quad (5)$$

$$\lambda \in [0, M].$$

Using (4), the EBM allows to estimate the amplitude A using two different procedures:

• **Direct procedure**

In this case the amplitude A is estimated as [1]:

$$\hat{A}_d = \frac{2}{M} \sqrt{\frac{\sum_{k=-H}^H |X_w(l+k)|^2}{NNPG}}, \quad (6)$$

where $NNPG$ is the window Normalized Noise Power Gain, defined as $NNPG = \sum_{m=0}^{M-1} w^2(m)/M$ [9]. For the H -term

cosine windows we have: $NNPG = a_0^2 + 0.5 \sum_{h=0}^{H-1} a_h^2$.

• **Indirect procedure**

By neglecting the effect of the image component of the spectrum, the amplitude A can be estimated as:

$$\hat{A}_i = \frac{2|X_w(l)|}{|W(-\hat{\delta})|}, \quad (7)$$

where $\hat{\delta}$ is the estimator of δ provided by the EBM [1]:

$$\hat{\delta} = \frac{\sum_{k=-H}^H k |X_w(l+k)|^2}{\sum_{k=-H}^H |X_w(l+k)|^2}. \quad (8)$$

3. ACCURACY COMPARISON OF THE EBM AMPLITUDE ESTIMATORS

The aim of this section is to evaluate and compare by means of computer simulations the sensitivity of the estimators \hat{A}_d and \hat{A}_i to the algorithm error, the spectral interference from the image component, and wideband noise. The most commonly used three-term ($H = 3$) and four-term ($H = 4$) cosine class windows are considered, that is [1], [7]:

- for $H = 3$: the maximum sidelobe decay window (*maxd3*), the rapid sidelobe decay 18 dB/octave window (*rsd18*), the minimum sidelobe level window (*minl3*), and the minimum error energy window (*mine3*);
- for $H = 4$: the maximum sidelobe decay window (*maxd4*), the rapid sidelobe decay 30 dB/octave window (*rsd30*), the minimum sidelobe level window (*minl4*), and the minimum error energy window (*mine4*).

The coefficients and some important features of these windows can be found in the scientific literature [1], [6], [7].

3.1. Effect of the algorithm error and the spectral interference

The algorithm error depends on δ , while the spectral interference mainly depends on both l and φ [5], [6]. Fig. 1 shows the magnitude of amplitude estimation errors achieved using both direct and indirect procedures, $|\Delta A_d|_{\max}$ and $|\Delta A_i|_{\max}$, as a function of l for the considered three-term (Fig. 1(a)) and four-term (Fig. 1(b)) cosine windows. The amplitude of the sine wave was $A = 2$ and the number of acquired samples was $M = 1024$. The integer part l was varied in the range $[2H + 2, 100]$, with a step of 2. For each value of l the fractional part δ took values in the range $[-0.5, 0.5)$ with a step of 0.01 and the phase φ was varied in the range $[0, 2\pi)$ rad with a step of $\pi/50$ rad. The errors $|\Delta A_d|_{\max}$ and $|\Delta A_i|_{\max}$ were then determined.

When a given three-term cosine window is considered, Fig. 1(a) shows that the errors introduced by the direct procedure are smaller than the errors associated to the indirect procedure, with the exception of the *rsd18* window. Also, Fig. 1(a) shows that, for small values of l , the highest accuracy is achieved when the direct procedure and the *mine3* window are used. Conversely, for high values of l , the best accuracy is provided by the indirect procedure and the *rsd18* window. Differently, using the considered four-term cosine windows, the direct procedure always exhibits a smaller uncertainty and the best accuracy is provided by the *mine4* window.

Other simulations have been performed for different values of the sine-wave amplitude A and the same behavior as in Fig. 1 was always achieved.

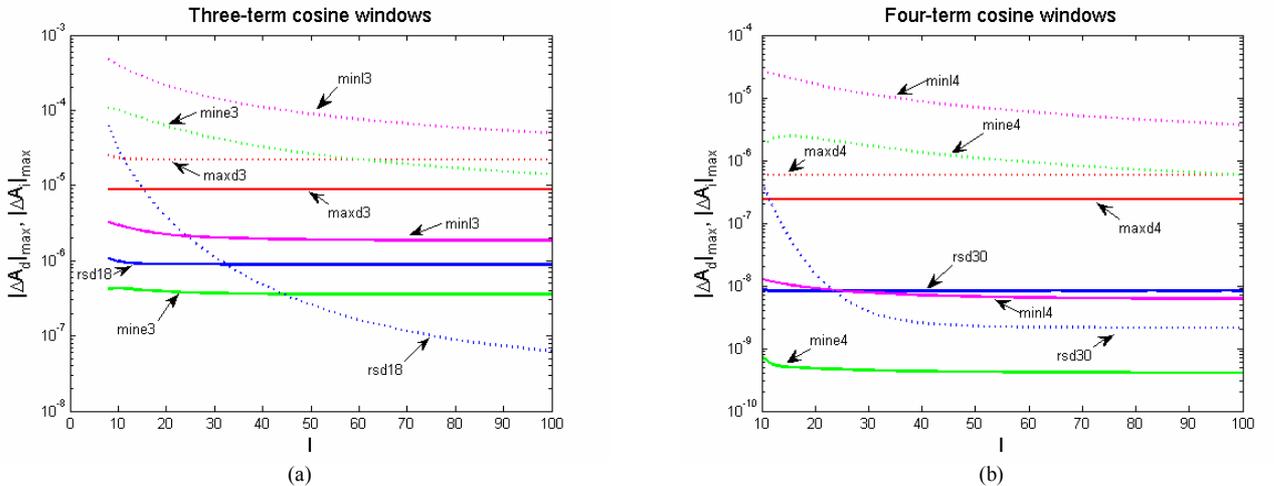


Fig. 1. Errors $|\Delta A_d|_{\max}$ and $|\Delta A_i|_{\max}$ versus the integer part l when three-term (a) and four-term (b) cosine windows are adopted. Both direct (continuous lines) and indirect (dotted lines) procedures are considered.

3.2. Effect of wideband noise

In order to model common real-life situations, we corrupted the sine wave using wideband additive Gaussian noise with zero mean and variance σ_n^2 . In any simulation run, both procedures exhibited a negligible estimation bias regardless of the adopted window.

Fig. 2 shows the behavior with respect to δ of the ratio between the variance of the estimators \hat{A}_d or \hat{A}_i and the corresponding Cramér-Rao Lower Bound (CRLB) [10]:

$$\text{var}[A]_{CR} \cong \frac{2\sigma_n^2}{M}. \quad (9)$$

The values achieved using both the three-term (Fig. 2(a)) and the four-term (Fig. 2(b)) cosine windows are considered.

The simulations parameters were: $A = 2$, $l = 93$, and $M = 1024$. The noise variance corresponds to a Signal-to-Noise Ratio (SNR) of 50 dB. The fractional part δ took values in the range $[-0.5, 0.5]$ with a step of $1/40$. For each value of δ , 5000 runs were performed and the sine wave phase φ was chosen at random in the range $[0, 2\pi)$ rad. Finally the variances of the amplitude estimates provided by both procedures were determined.

It should be noticed that for the used values of l the magnitudes of the amplitude estimation errors due to the algorithm error and the spectral interference (see Fig. 1) are negligible with respect to the effect of wideband noise for both procedures.

Moreover, Fig. 2 shows that using suitable windows the indirect procedure exhibits a lower estimator variance than the direct procedure for both three- and four-term cosine windows.

Concluding, it should be remarked that, the overall accuracy of both amplitude estimation algorithms is limited by the joint effects of the algorithm error, the spectral interference by the image component, and wideband noise. However, in practical applications the effect of noise usually prevails when suitable three-term or four-term cosine class windows are employed.

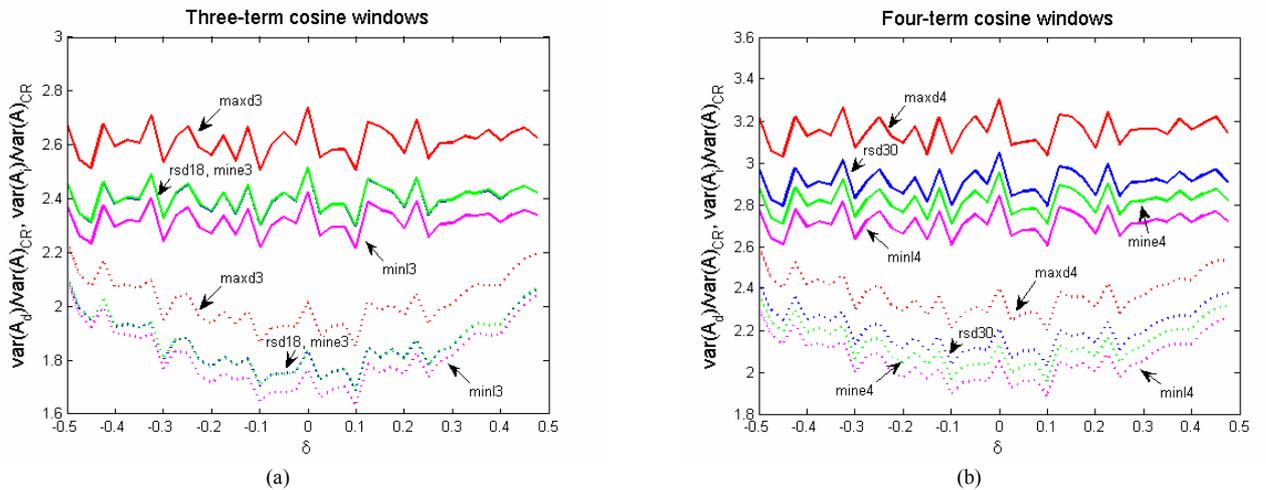


Fig. 2. Ratio between the variance of the estimator \hat{A}_d or \hat{A}_i and the corresponding CRLB versus the fractional part δ when three-term (a) and four-term (b) cosine windows are adopted. Both direct (continuous lines) and indirect (dotted lines) procedures are considered.

4. EXPERIMENTAL RESULTS

The overall accuracies of the considered amplitude estimation algorithms have been compared using experimental results too.

In the experimental runs the sine waves were supplied by an Agilent 33220A signal generator and acquired by a 12-bit data acquisition board NI-6023E, developed by National Instruments. The Full Scale Range, FSR , of the converter and the sampling frequency were set to 10 V and 100 kHz, respectively.

In Fig. 3 variances of the estimators \hat{A}_d (Fig. 3(a)) and \hat{A}_i (Fig. 3(b)) are depicted as a function of the fractional part δ when the three-term cosine windows are adopted. The same results are depicted in Fig. 4, but when the four-term cosine windows are adopted. The amplitude of the signals was set to 4 V and the related frequencies were varied between 9.09 and 9.17 kHz with a step of 10 Hz in order to obtain different values of δ in the range $(-0.5, 0.5)$. For each value of frequency 5000 runs of $M = 1024$ samples each were performed and the variances of the estimators \hat{A}_d and \hat{A}_i were calculated. The fractional part δ was estimated by using the mean value of the estimates returned by the EBM applied with the *rsd18* window.

Fig. 3 shows that the variances of estimators \hat{A}_i are higher when the *minl3* window is adopted. Indeed the effect of the algorithm error and the spectral interference is significant in that case, as we can deduce from Fig. 1. The behavior of all the other results is the same as in simulations. Thus, once more we can conclude that the indirect procedure exhibits a lower estimation uncertainty than the direct procedure.

5. CONCLUSION

In the paper the performances of the amplitude estimators provided by the EBM using both direct and indirect procedures have been compared when the most commonly cosine class windows are adopted.

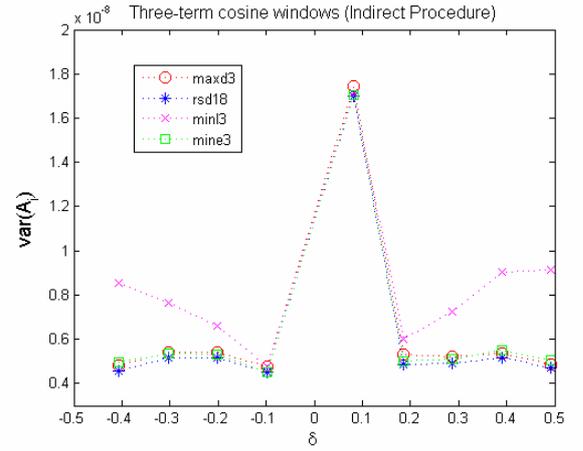
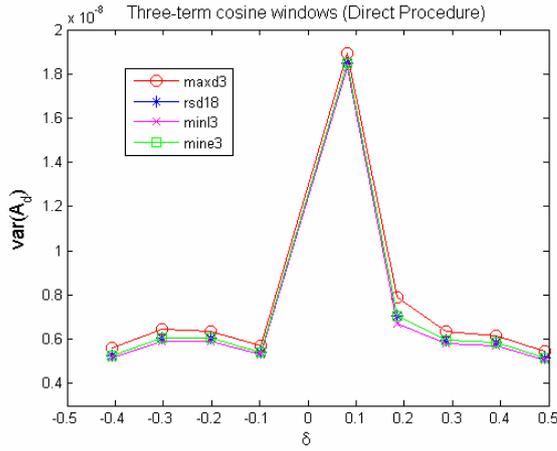


Fig. 3. Variances of the estimators \hat{A}_d (a) and \hat{A}_i (b) versus δ achieved by means of experimental results when the three-term cosine windows are adopted.

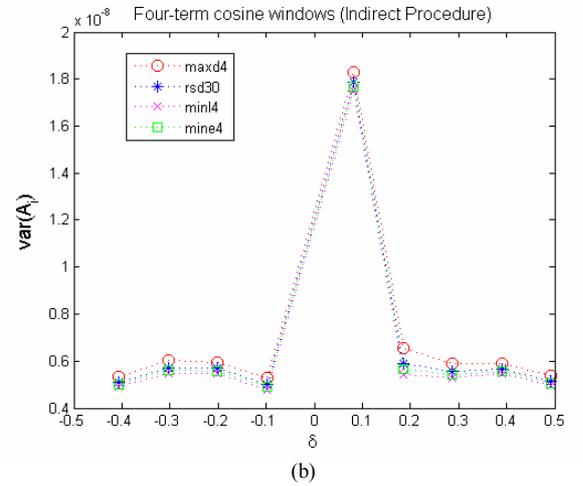
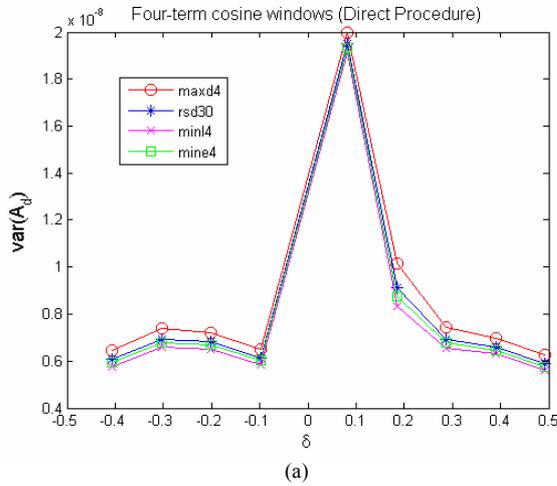


Fig. 4. Variances of the estimators \hat{A}_d (a) and \hat{A}_i (b) versus δ achieved by means of experimental results when the four-term cosine windows are adopted.

It has been shown that when three-term cosine windows are adopted, if the integer part of the acquired sine wave cycles, l , is quite small, the direct procedure exhibits the smallest sensitivity to the algorithm error and the spectral interference from the image component when the *mine3* window is adopted. Conversely, the indirect procedure provides the best accuracy if a quite large number of sine wave cycles is acquired and the *rsd18* window is employed. Using four-term cosine windows, the minimum sensitivity to the algorithm error and the spectral interference is achieved by the direct procedure based on the *mine4* window.

Moreover, it has been shown that the amplitude estimates achieved by indirect procedure are less sensitive to wideband noise than those provided by the direct procedure. The results reported in the paper are validated by means of experimental results.

REFERENCES

[1] C. Offelli and D. Petri, "A Frequency-Domain Procedure for Accurate Real-Time Signal Parameter Measurement", *IEEE Trans. on Instrum. and Meas.*, vol. 39, no. 2, pp. 363-368, April 1990.

[2] O. M. Solomon, "The Use of DFT Windows in Signal-to-Noise Ratio and Harmonics Distortion Computations", *IEEE Trans. on Instrum. and Meas.*, vol. 43, no. 2, pp. 194-199, April 1994.

[3] D. Petri, "Frequency-Domain Testing of Waveform Digitizers", *IEEE Trans. Instrum. and Meas.*, vol. 51, no. 3, pp. 445-453, June 2002.

[4] M. Novotný, D. Slepíčka, and M. Sedláček, "Uncertainty Analysis of the RMS Value and Phase in Frequency Domain by Noncoherent Sampling", *IEEE Trans. on Instrum. and Meas.*, vol. 56, no. 3, pp. 983-989, June 2007.

[5] D. Belega, D. Dallet, and D. Petri, "Optimal Windows for Sine-Wave Amplitude Estimation by the Energy-Based Method", 17th Symposium IMEKO TC 4, September 8-10, Kosice, Slovakia, 2010.

[6] D. Belega, D. Dallet, and D. Petri, "Accuracy of the Normalized Frequency Estimation of a Discrete-Time Sine-Wave by the Energy-Based Method", in publication.

[7] A. H. Nuttall, "Some Windows with Very Good Sidelobe Behavior", *IEEE Transactions on Acoustics, Speech, and Signal Processing*, Vol. ASSP-29, no.1, pp 84-91, February 1981.

[8] D. Belega and D. Dallet, "Multifrequency Signal Analysis by Interpolated DFT Method with Maximum Sidelobe Decay Windows," *Measurement*, vol. 42, no. 3, pp. 420-426, April 2009.

[9] F. J. Harris, "On the Use of Windows for Harmonic Analysis with the Discrete Fourier Transform", *Proceedings of the IEEE*, vol. 66, pp 51-83, January 1978.

[10] C. Offelli and D. Petri, "The Influence of Windowing on the Accuracy of Multifrequency Signal Parameter Estimation", *IEEE Trans. Instrum. and Meas.*, vol. 41, no. 2, pp. 256-261, April 1992.