PARAMETER ESTIMATION EMPLOYING A DUAL CHANNEL SINE-WAVE MODEL UNDER SPARSE COPRIME SAMPLING

Senay Negusse[1] *and Peter Händel*[2]

[1] Royal Institute of Technology, ACCESS Linnaeus Center, Signal Processing Lab, Stockholm, Sweden, *negusse@kth.se*

[2] Royal Institute of Technology, ACCESS Linnaeus Center, Signal Processing Lab, Stockholm, Sweden, *ph@kth.se*

Abstract: In this paper, we consider a dual channel signal model with common frequency that is appropriate in a plurality of measurement setups, including impedance measurements and imbalance measurement in direct conversion RF receivers. We investigate on a new approach and present analysis of parameter estimation, specifically the seven parameter dual channel sine fit in which the signals from both channels are uniformly subsampled with coprime pair of sparse samplers and combined in a single array. Simulation results shall show acquired improvement in complexity as well as loss in estimation performance compared to that of an estimation from Nyquist sampled signal.

1. INTRODUCTION

Demands on sensors, instruments and intrinsic measurement devices, such as analog to digital converters (ADCs), are mostly related to the sampling frequency required for the application at hand. As the current trend for testing and verification of devices is demanding for higher bandwidth, there is a need for schemes based on undersampling, employing the sparseness of the analyzed data.

The problem formulated in this paper is categorized among the set of problems dealt with in compressive sensing, or compressive sampling. The underlaying study in compressive sensing/sampling introduces approaches to sensing and acquiring signals based on sampling at rates significantly below the Nyquist rate. The theory developed thus far has made it possible to combine sub-Nyquist sampling with current available computational power for solving estimation problems efficiently. A number of advantages have been pointed out to employing compressive sensing; one major outcome has been to significantly reduce the sampling rate required by analog-to-digital converters for wideband applications [3], [4].

The well known sin-wave-fit algorithm defined e.g. by IEEE Standard 1057 [1] has provided an essential tool for testing of ADCs, waveform recorders and communication electronics. The method extract the in-phase and quadrature components of the sine-wave, the direct-current (DC), and (for the, so-called, four parameter fit) also the angular frequency of the sine-wave. See e.g. [2] for details on its performance.

Dual channel measurements relying on the sinewave model are common, e.g. in applications like laser anemometry [5], impedance measurements [6], and characterization of mixers in radio-frequency receivers [7]. In [8], the dual channel model was studied and an extension of the four-parameter fit in [1] to a seven-parameter fit was proposed.The Gaussian scenario was studied in some detail in [9]; including the Cramér-Rao bound (CRB) for performance assessments, and non-linear least-squares for parameter estimation. Asymptotic results were derived and compared with numerical results based on limited set of samples indicating good agreement between theory and practice.

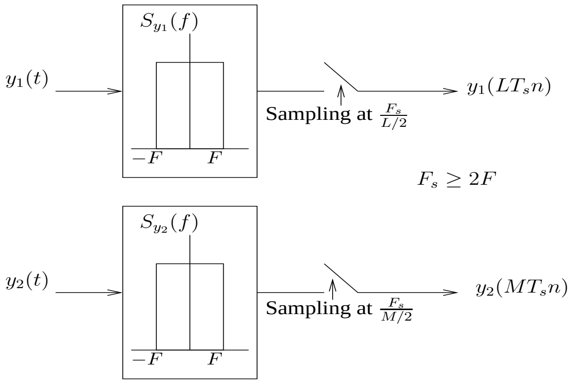

Fig. 1. Basic set-up. Compared with Nyquist sampling of the channels, the channels are down-sampled by $1/2$ of the coprime factors L and M, respectively.

The purpose of this paper is to apply some ideas of compressive sensing to the considered dual channel sine-wave model. In particular, the seven parameter dual channel model is studied under coprime subsampling; See Fig. 1 for the basic set-up and notation. Employing coprime subsampling corresponds to a reduction of the sampling rate by a factor $L/2$ in the first channel, and a factor $M/2$ in the second, where $L > 2$ and $M > 2$ are coprime integers. To use coprime subsampling of the channels was proposed in a similar context in [10], [13]–[15]. Among other things it was shown in [10] that dual channel coprime sampling allows to sample a signal sparsely and estimate some aspects of the signal at a significantly higher resolution.

The main motivation for the paper is the fact that the sought for signal parameters can be uniquely determined, by a clever choice of under-sampling strategy with the benefit of a $\min(L/2, M/2)$-factor reduction of the sampling rate. A result of the sampling strategy is that (1) for a given resolution of the front-end ADC, a lower cost unit can be employed, (2) for a given cost of the ADC, the resolution can be increased, leading to improved performance, and (3) by a reduced rate the integration time in the ADC can be extended, leading to superior signal-to-noise ratios. The price paid is for a given measurement time, less amount of data is collected leading to reduced estimation accuracy, however it may be motivated by a reduced implementation complexity. We will apply the methodology of [9] and consider the problem at hand under the Gaussian assumption, and resolve some fundamental questions on estimation accuracy and parameter estimation, combined with some simulation results. The provided analysis is conservative, because the benefits above are not considered, but can be analyzed one-by-one by the given methodology.

The organization of the paper is as follows: Section 2 presents the employed dual channel signal model upon which the problem is formulated and the computed CRB for coprime subsampling implementation is shown in Section 3. Section 4 discusses on proposed technique for estimating the common frequency including nonlinear least squares (NLS) methods for the seven parameter subsampled sine-fit. In Section 5, we present simulation results and analysis for the practical case in testing of direct conversation RF-receivers for IQ-imbalance problem in RF receivers based employing coprime subsampling. we draw our conclusion in Section 6.

2. PROBLEM FORMULATION

First the Nyquist sampled model is reviewed, followed by introducing the model corresponding to sub-Nyquist sampling.

2.1. Signal model

For easy reference, the dual-channel (Nyquist sampled) signal model is given by the measurements collected in the measurement vectors

$$\mathbf{y}_1 = (y_1(0), \ldots, y_1(N-1))^T \tag{1}$$

and

$$\mathbf{y}_2 = (y_2(0), \ldots, y_2(N-1))^T \tag{2}$$

where the index $\{1, 2\}$ denotes the channel number, and N the number of samples per channel. Further, T denotes the transpose operator. An additive model of the measurements is introduced as

$$y_k(n) = s_k(n) + v_k(n), \quad k = 1, 2, \quad n = 0, \ldots, N-1 \tag{3}$$

where $v_k(n)$ is white Gaussian noise with variance σ_k^2, jointly independent between the channels (that is, circular white noise). Further, the signal of interest is modeled by (for $n = 0, \ldots, N-1$)

$$s_k(n) = A_k \cos(\omega n) + B_k \sin(\omega n) + C_k, \quad k = 1, 2 \tag{4}$$

where A_1 and B_1 are the amplitudes of the in-phase and quadrature components, and C_1 is the DC component for the first channel; while A_2, B_2 and C_2 are the corresponding parameters for the second channel. The angular frequency $\omega = 2\pi F/F_s$, where F is the absolute frequency in Hertz (Hz), is common to both channels as can be seen from (4). For the problem at hand, the sought for parameters are gathered in the vector:

$$\theta = \begin{pmatrix} A_1 & B_1 & C_1 & \omega & A_2 & B_2 & C_2 \end{pmatrix}^T \tag{5}$$

The signal model (1)–(5) has been studied in [9]. There exist a variety of algorithms for parameter extraction, such as iterative non-linear least-squares methods [8, 9] or ellipse fitting [11].

2.2. Coprime sampling

Starting with the measurements in (1)–(2) the subsampled data, where $\mathbf{y}_1$ and $\mathbf{y}_2$ are sub-sampled by half of the coprime factors L and M respectively, is given by

$$\mathbf{y}_L = (y_1(0),\, y_1(L/2), \ldots, y_1((N-1)L/2))^T \tag{6}$$

and

$$\mathbf{y}_M = (y_2(0),\, y_2(M/2), \ldots, y_2((N-1)M/2))^T \tag{7}$$

We emphasize that the subsampling of a longer data vector is only introduced for notational convenience. In a practical set-up, measurements are directly collected employing a reduced sampling rate. For notational convenience we have further denoted the sub-set of $\mathbf{y}_1$ by $\mathbf{y}_L$, which is a vector of length $N_1 = 2N/L$, and sub-set of $\mathbf{y}_2$ by $\mathbf{y}_M$, which is a vector of length $N_2 = 2N/M$. The integers L and M are the integer coprime subsampling factors. By requirement, N_1 and N_2 have to be integers, which can be assured by proper selection of N.

Due to the different sampling rates, $2F_s/L$ and $2F_s/M$, the angular frequency of the sinewave components are no longer equal. In addition, the subsampling may result in a folded component (aliasing) which will present an ambiguity when trying to resolve the frequency estimates. Let ν_L and ν_M denote the (possibly) folded angular frequency in channel 1 and channel 2 respectively, with $\nu_L, \nu_M \in [0,\, \pi)$. Then

$$\nu_L = \frac{L}{2}\,\omega \text{ modulo } \pi \tag{8}$$

and

$$\nu_M = \frac{M}{2}\,\omega \text{ modulo } \pi \tag{9}$$

For example, with $\omega = \pi/8$ for $L = 3$ and $M = 4$, then $\nu_L = 3\pi/16$ and $\nu_M = \pi/4$, respectively and from which the parameter ω is to be determined.

3. CRAMÉR-RAO BOUND

The CRB is a lower bound on the error variance of any unbiased estimator [17]. Under the Gaussian assumption,

Table 1 . Comparing CRB for the estimation the seven parameters using standard Nyquist sampling and using the employed Coprime sampling.

	Nyquist sampling	Coprime sampling
A_1	$\frac{1}{N}\left(2\sigma_1^2 + \frac{3B_1^2}{\text{SNR}}\right)$	$\frac{L}{2N}\left(2\sigma_1^2 + \frac{3B_1^2}{(\text{SNR}_1+\frac{L^4}{M^4}\text{SNR}_2)}\right)$
A_2	$\frac{1}{N}\left(2\sigma_i^2 + \frac{3B_2^2}{\text{SNR}}\right)$	$\frac{M}{2N}\left(2\sigma_2^2 + \frac{3B_2^2}{(\frac{M^4}{L^4}\text{SNR}_1+\text{SNR}_2)}\right)$
B_1	$\frac{1}{N}\left(2\sigma_1^2 + \frac{3A_1^2}{\text{SNR}}\right)$	$\frac{L}{2N}\left(2\sigma_1^2 + \frac{3A_1^2}{(\text{SNR}_1+\frac{L^4}{M^4}\text{SNR}_2)}\right)$
B_2	$\frac{1}{N}\left(2\sigma_2^2 + \frac{3A_2^2}{\text{SNR}}\right)$	$\frac{M}{2N}\left(2\sigma_2^2 + \frac{3A_2^2}{(\frac{M^4}{L^4}\text{SNR}_1+\text{SNR}_2)}\right)$
C_1	$\frac{\sigma_1^2}{N}$	$\frac{L\sigma_1^2}{2N}$
C_2	$\frac{\sigma_2^2}{N}$	$\frac{M\sigma_2^2}{2N}$
ω	$\frac{12}{N^3(\text{SNR})}$	$\frac{12L}{2N^3(\text{SNR}_1+\frac{L^4}{M^4}\text{SNR}_2)}$

the CRB for the Nyquist sampled model (1)–(5) was derived in [9]. The methodology in [9] can be employed for the situation at hand, *mutatis mutandis*. With coprime subsampling, the signal model in (4) is modified to

$$s_k(\alpha_k n) = A_k\cos(\alpha_k\omega\, n)+B_k\sin(\alpha_k\omega\, n)+C_k,\ k=1,2 \quad (10)$$

with $\alpha_k \in \{L/2,\, M/2\}$. The CRB is derived based on the subsampled signal model (10). Let

$$\mathbf{y} = (\mathbf{y}_L\ \mathbf{y}_M)^T \quad (11)$$

where $\mathbf{y}_L$ and $\mathbf{y}_M$ are given by (6) and (7) respectively. The derivation of the CRB involves inverting the 7×7 Fisher information matrix given below

$$\mathbf{J}(\theta) = E\left[\left(\frac{\partial \ln p(\mathbf{y};\theta)}{\partial\theta}\right)\left(\frac{\partial \ln p(\mathbf{y};\theta)}{\partial\theta}\right)^T\right] \quad (12)$$

where $p(\mathbf{y};\theta) = p(\mathbf{y_L};\theta)\cdot p(\mathbf{y_M};\theta)$, which is multiple of two Gaussian pdfs with variances σ_1^2 and σ_2^2 respectively. Computing (12), the elements in the matrix $\mathbf{J}(\theta)$ are given by

$$\begin{aligned}[\mathbf{J}(\theta)]_{i,j} = &\frac{1}{\sigma_1^2}\sum_{n=0}^{N_1-1}\frac{\partial s_1(\frac{L}{2}n;\theta)}{\partial\theta_i}\frac{\partial s_1(\frac{L}{2}n;\theta)}{\partial\theta_j}\\ &+\frac{1}{\sigma_2^2}\sum_{n=0}^{N_2-1}\frac{\partial s_2(\frac{M}{2}n;\theta)}{\partial\theta_i}\frac{\partial s_2(\frac{M}{2}n;\theta)}{\partial\theta_j}\end{aligned} \quad (13)$$

Asymptotic CRB is derived by taking the inverse the matrix $\mathbf{J}(\theta)$ using similar method and applying the simplifying properties for large enough N employed in [9] and [12].

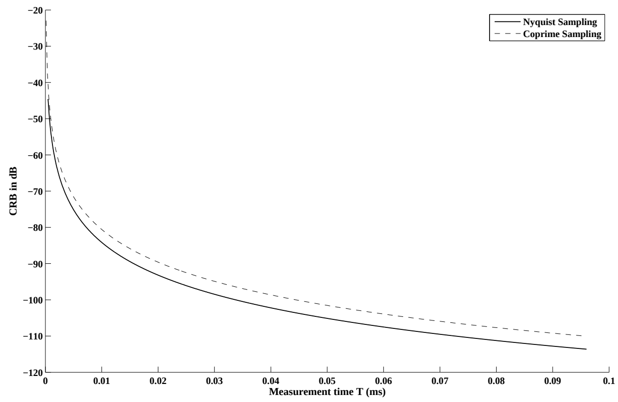

Fig. 2. CRB of ω for varying measurement time T employing Nyquist sampling (solid line) and Co-prime subsampling (dots). Here, $L = 3$ and $M = 4$.

In Table 1, the CRBs of the seven parameter model is shown, comparing Nyquist sampling and the employed coprime sampling strategy for a given signal-to-noise ratio (SNR). The SNR is defined as

$$\text{SNR} = \text{SNR}_1 + \text{SNR}_2$$

where $\text{SNR}_k = (A_k^2 + B_k^2)/2\sigma_k^2$ for $k = 1, 2$. The table presents the obvious outcome in estimation performance under subsampling, that is for a given measurement time T the number of acquired samples will be less than the number of samples collected in Nyquist sampling, consequently making the CRB larger. A direct observation is that if both channels were to be subsampled with the same factor such that $L = M$, then the CRB for all parameters obtained from Nyquist sampling would increase by a factor of $L/2$ or $M/2$. In general however, when L and M are c oprime, the CRB still increases but by a non integral factor which is a function of L and M as shown in Figure 2 for ω.

4. FREQUENCY ESTIMATION

In this section, a method to estimate the angular frequency is discussed. Noting that due to subsampling each channel, which possibly renders one of or both waveforms to be undersampled, it is impossible to uniquely determine the frequency from only one of the subsampled channels. However, as presented in [10], [13]–[15] determination of frequencies from multiple undersampled waveforms is possible. [13] showed the Chinese Remainder Theorem (CRT) for estimation of a single frequency in complex valued undersampled waveform, while [15] introduced an algorithm employing real valued waveform corresponding to two symmetric frequencies in a complex valued waveform which also corresponds to the set-up in this paper. In this paper, we form four set of equations based on frequency estimates of each subsampled channel from which the true common frequency ω is to be resolved.

4.1. Resolving the frequency ambiguity

The relations in (8)–(9) present an $(LM)/2$-ambiguity in ω within $[0, \pi)$ which has to be resolved to uniquely determine its estimate. Denote initial estimates of the frequency by $\widehat{\nu}_L$ and $\widehat{\nu}_M$, respectively, both belonging to the interval $[0, \pi)$. Because of the real valued signal model, the ambiguity extends such that each frequency estimate from the sine-fit algorithm is paired with its conjugate, $\widehat{\nu}_L^* = 2\pi - \widehat{\nu}_L$ and $\widehat{\nu}_M^* = 2\pi - \widehat{\nu}_M$, in the interval $(\pi, 2\pi]$. The paired estimates $\{\widehat{\nu}_L, \widehat{\nu}_L^*\}$ and $\{\widehat{\nu}_M, \widehat{\nu}_M^*\}$ are related to the true frequency ω by (8)–(9). Therefore, for some integers ℓ and m, where $\ell = 0, \dots, L/2 - 1$ and $m = 0, \dots, M/2 - 1$ the following sets of candidate estimates are computed

$$\begin{aligned} \widehat{\omega}_{\ell,1} &= 2(\widehat{\nu}_L + \ell\pi)/L \\ \widehat{\omega}_{\ell,2} &= 2(\widehat{\nu}_L^* + \ell\pi)/L \end{aligned} \tag{14}$$

(for $\ell = 0, \dots, \frac{L}{2} - 1$) and

$$\begin{aligned} \widehat{\omega}_{m,1} &= 2(\widehat{\nu}_M + m\pi)/M \\ \widehat{\omega}_{m,2} &= 2(\widehat{\nu}_M^* + m\pi)/M \end{aligned} \tag{15}$$

(for $m = 0, \dots, \frac{M}{2} - 1$). The solution $\widehat{\omega}$ is then obtained by choosing

$$\ell \in \left\{0,\, 1,\, \cdots,\, \frac{L}{2} - 1\right\}, m \in \left\{0,\, 1,\, \cdots,\, \frac{M}{2} - 1\right\}$$

and $i, j \in \{1,\, 2\}$ for the pair $(\widehat{\omega}_{\ell,i},\, \widehat{\omega}_{m,j})$, such that

$$\widehat{\ell},\, \widehat{m},\, \widehat{i},\, \widehat{j} = \arg\min_{\ell,m,i,j} |\widehat{\omega}_{\ell,i} - \widehat{\omega}_{m,j}| \tag{16}$$

Equation (16) will have a unique solution $\widehat{\omega} \in [0, \pi)$. In the noise free case, it follows that when L and M are coprime the relation below holds [15]

$$\min_{\ell,m,i,j} |\widehat{\omega}_{\ell,i} - \widehat{\omega}_{m,j}| = 0$$

We therefore have the frequencies $\widehat{\omega}_{\widehat{\ell},\widehat{i}}$ and $\widehat{\omega}_{\widehat{m},\widehat{j}}$ which minimize (16) and from which the final estimate $\widehat{\omega}$ is estimated by averaging as below

$$\widehat{\omega} = \frac{1}{2}\left(\widehat{\omega}_{\widehat{\ell},\widehat{i}} + \widehat{\omega}_{\widehat{m},\widehat{j}}\right) \tag{17}$$

The two examples below try to elaborate the discussion for any solution, $(\widehat{\omega}_{\widehat{\ell},\widehat{i}},\, \widehat{\omega}_{\widehat{m},\widehat{j}})$ such that $\widehat{\omega}_{\widehat{\ell},\widehat{i}} = \widehat{\omega}_{\widehat{m},\widehat{j}} = \widehat{\omega}$, in the noise free case.

Example 1: Consider the example with $\widehat{\nu}_L = 15\pi/16$ and $\widehat{\nu}_M = 3\pi/4$ for $L = 3/2$ and $M = 4/2$, corresponding to a sought for frequency $\omega = 5\pi/8$. Then

$$\widehat{\omega}_{\ell,1} \in \frac{\pi}{8}\{5\} \tag{18}$$

$$\widehat{\omega}_{\ell,2} \in \frac{\pi}{24}\{17\} \tag{19}$$

$$\widehat{\omega}_{m,1} \in \frac{\pi}{8}\{3,\, 7\} \tag{20}$$

$$\widehat{\omega}_{m,2} \in \frac{\pi}{8}\{5,\, 9\} \tag{21}$$

Solving (16) results in the two solutions $(\widehat{\ell},\, \widehat{m},\, \widehat{i},\, \widehat{j})$ given by $(0,\, 0,\, 1,\, 2)$. In which the frequency estimates is given by $\widehat{\omega} = 5\pi/8$.

Example 2: Consider also the example with $\widehat{\nu}_L = 9\pi/16\pi$ and $\widehat{\nu}_M = 3\pi/4$ for $L = 3/2$ and $M = 4/2$, corresponding to a sought for frequency $\omega = 3\pi/8$.

$$\widehat{\omega}_{\ell,1} \in \frac{\pi}{8}\{3\} \tag{22}$$

$$\widehat{\omega}_{\ell,2} \in \frac{\pi}{24}\{23\} \tag{23}$$

$$\widehat{\omega}_{m,1} \in \frac{\pi}{8}\{3,\, 7\} \tag{24}$$

$$\widehat{\omega}_{m,2} \in \frac{\pi}{8}\{5,\, 9\} \tag{25}$$

once again solving (16) results in the solutions given by $(0,\, 0,\, 1,\, 1)$.

4.2. Initializing with dual four-parameter fits

The four-parameter fit, [1], [2] is applied once per channel to obtain estimate of the frequencies $\hat{\nu}_L$ and $\hat{\nu}_M$ from the subsampled waveforms. Equations (14)–(17) are further employed on the two individual initial estimates to obtain an unambiguous initial estimate of the common frequency $\widehat{\omega}^0$, around which a numerical maximization procedure is to be employed in the non-linear least squares problem discussed in the following section.

4.3. Non-Linear Least Squares

A nonlinear least squares framework was provided in [9] for solving the dual channel seven-parameter sine fit in which Gaussian noise with known variance was assumed for each channel. Following similar notations, we form a generalization to solving the nonlinear least squares criterion for coprime subsampled channels. The least squares criterion given below

$$\begin{aligned} V_N(\theta) = &\sum_{n=0}^{N_L-1} (y_L(n) - s_L(n;\, \theta))^2 + \\ &\sum_{n=0}^{N_M-1} (y_M(n) - s_M(n;\, \theta))^2 \end{aligned} \tag{26}$$

where $N_L = 2N/L$, $N_M = 2N/M$ and

$$\begin{aligned} s_L(n;\, \theta) &= A_1 \cos(\frac{L}{2}\omega\, n) + B_1 \sin(\frac{L}{2}\omega\ n) + C_1, \\ s_M(n;\, \theta) &= A_2 \cos(\frac{M}{2}\omega\, n) + B_2 \sin(\frac{M}{2}\omega\, n) + C_2 \end{aligned} \tag{27}$$

noting that the models $s_L(n;\, \theta)$ and $s_M(n;\, \theta)$ are equivalent to the coprime subsampled signal model in (10). The least squares problem therefore reads

$$\widehat{\theta} = \arg\min_{\beta} V_N(\theta) \tag{28}$$

in order to continue, let us define matrices $\mathbf{d}_L$ and $\mathbf{d}_M$ along with a block matrix $\mathbf{D}$ such as given below

$$\mathbf{d}_k = \begin{pmatrix} \cos\frac{k}{2}\omega\cdot 0 & \sin\frac{k}{2}\omega\cdot 0 & 1 \\ \cos\frac{k}{2}\omega\cdot 1 & \sin\frac{k}{2}\omega\cdot 1 & 1 \\ \vdots & \vdots & \vdots \\ \cos\frac{k}{2}\omega\cdot (N_k-1) & \sin\frac{k}{2}\omega\cdot (N_k-1) & 1 \end{pmatrix}. \tag{29}$$

and

$$\mathbf{D} = \begin{pmatrix} \mathbf{d}_L & \mathbf{0} \\ \mathbf{0} & \mathbf{d}_M \end{pmatrix} \tag{30}$$

where $\mathbf{0}$ is a zero matrix of appropriate size and k denotes the coprime subsampling factors L and M and $N_k = 2N/k$. Therefore, (26) can be put in a matrix form as shown below

$$V_N(\theta) = (\mathbf{y} - \mathbf{D}\,\mathbf{x})^T(\mathbf{y} - \mathbf{D}\,\mathbf{x}). \tag{31}$$

where $\mathbf{x}$ is the parameter vector with the linear parameters, see [9]. By condensing the loss function $V_N(\theta)$ with respect to the linear parameters, a maximization problem given below remains [9]

$$\hat{\omega} = \arg\max_{\omega} g(\omega) \tag{32}$$

where $g(\omega)$ is given as

$$\begin{aligned} g(\omega) &= \mathbf{y}^T\mathbf{D}(\mathbf{D}^T\mathbf{D})^{-1}\mathbf{D}^T\mathbf{y} \\ &= g_L(\omega) + g_M(\omega). \end{aligned} \tag{33}$$

such that

$$\begin{aligned} g_L(\omega) = \mathbf{y}_L^T\mathbf{d}_L(\mathbf{d}_L^T\mathbf{d}_L)^{-1}\mathbf{d}_L^T\mathbf{y}_L \\ g_M(\omega) = \mathbf{y}_M^T\mathbf{d}_M(\mathbf{d}_M^T\mathbf{d}_M)^{-1}\mathbf{d}_M^T\mathbf{y}_M \end{aligned} \tag{34}$$

As earlier noticed, coprime subsampling introduces an ambiguity in the determination of the underlying angular frequency. Therefore, having acquired the initial and unambiguous estimate $\widehat{\omega}^0$, the final estimate, $\hat{\omega}$, is obtained using numerical maximization of (32) around the initial value with the iterative solution provided in [8] or with the aid of the scalar bounded nonlinear function minimization routine *fminbnd* provided by MATLAB as suggested in [9] and from which simulation results are provided in the next section.

4.4. Linear parameters

The criterion defined in (26) is within the set of separable nonlinear least squares problems [17] in which it is separated into two subproblems that depend on the linear and non-linear parameters respectively. Having obtained estimate of the nonlinear parameter ω from the subsampled set signals, the parameter vector

$$\mathbf{x} = (A_1,\, B_1,\, C_1,\, A_2,\, B_2,\, C_2)^T$$

which contains the six linear parameters is estimated by solving the least-squares solution

$$\begin{aligned} \widehat{\mathbf{x}} &= (\mathbf{D}^T\mathbf{D})^{-1}\mathbf{D}^T\mathbf{Y}_{LM} \\ &= \begin{bmatrix} (\mathbf{d}_L^T\mathbf{d}_L)^{-1}\mathbf{d}_L^T\mathbf{Y}_L \\ (\mathbf{d}_M^T\mathbf{d}_M)^{-1}\mathbf{d}_M^T\mathbf{Y}_M \end{bmatrix}. \end{aligned} \tag{35}$$

One may note that (35) is equivalent to employing the dual three-parameter sinewave fits.

5. APPLICATION AND SIMULATION RESULTS

Subsampling may for example be used to speed-up test during the production phase. Massive parallel testing of a large number of receivers put demands on efficient testing methods. Coprime subsampling loosens the requirement on memory and processing power to a moderate cost, and may therefore be the method of choice for testing and determination of IQ-imbalance in direct conversion RF-receivers.

5.1. Estimation of IQ-Imbalance parameters

IQ imbalance is among the multiples of impairments that occurs in radio frequency (RF) receivers' circuitry. For direct conversion receivers where signal in the RF band is translated directly to baseband frequency, IQ-imbalance causes distortion of the IQ-signals leading to degradation of the reception performance. In [7], a single tone test for estimation of IQ imbalance parameters was employed. The problem was parameterized in such a way that it allowed for a nonlinear least squares approach for the estimation of parameters. modeling of the noise free signals in the I-branch and the Q-branch of a direct-conversion receiver is given by

$$\begin{aligned} s_I(n) &= g_I\cos(\omega n + \phi_I) + C_I \\ &= \sqrt{A_I^2 + B_I^2}\cos\left(\omega\, n + \arctan\frac{A_I}{B_I}\right) + C_I. \\ &= A_I\cos\left(\omega n + \frac{\pi}{2}\right) + B_I\sin\left(\omega n + \frac{\pi}{2}\right) + C_I. \end{aligned} \tag{36}$$

and

$$\begin{aligned} s_Q(n) &= g_Q\sin(\omega n + \phi_Q) + C_Q. \\ &= \sqrt{A_Q^2 + B_Q^2}\sin\left(\omega\, n + \arctan\frac{A_Q}{B_Q}\right) + C_Q \\ &= A_Q\cos(\omega n) + B_Q\sin(\omega n) + C_Q. \end{aligned} \tag{37}$$

where the parameters g_I and g_Q are gains, ϕ_I and ϕ_Q are initial phases and C_I and C_Q are DC offsets in each branch. After employing coprime subsampling, the measured signal samples from each branch can be put in the form of (2) assuming gaussian distributed measurement noise for implementation of the seven parameter sine-fit from subsampled pair of signals (I branch is equivalent to channel 1 and Q branch is equivalent to channel 2). Typically, the gain imbalance $G = \frac{g_I}{g_Q}$, the quadrature skew $Q = \phi_I - \phi_Q$ and the DC leakage in the local oscillator

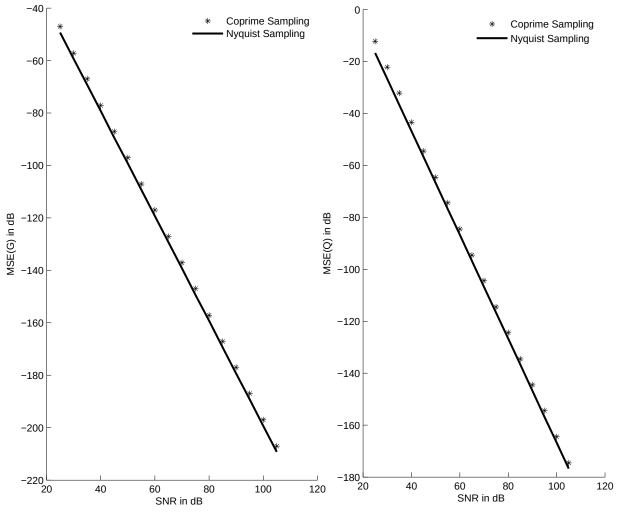

Fig. 3. Mean Squared Error in estimating Gain imbalance, $\hat{G}_{dB}$ (Left) and Quadrature skew $\hat{Q}_o$ (Right), employing seven parameter fit under Nyquist sampling (solid line) and co-prime sampling (dots). Here, $L = 3$ and $M = 4$ for $F_s = 20MHz$, $T = 6\mu sec$ with $N = 120$ and 10000 simulation runs.

$L = 2\frac{C_I^2+C_Q^2}{g_I^2+g_Q^2}$ characterize the distortion introduced by IQ-imbalance and which also are parameters to be determined from the primarily estimated parameters using the seven parameter sine fit procedure. By significantly reducing the required sampling frequency (Nyquist frequency) for estimating the parameters, the employed coprime subsampling technique should ease the demand on the ADC of RF receiver electronics.

Creating identical simulation environment as in [7], the plots in Figure 2 show comparison of estimation performance of $\hat{G}$, and $\hat{Q}$ and $\hat{L}$ in terms of mean squared error employing Nyquist sampling and coprime subsampling for varying number of samples N. The coprime subsampling factors in the I and Q branch are taken to be $L = 3$ and $M = 4$ for an $\omega \in [0, \pi/2]$ respectively. The sampling frequency is greatly reduced at the expense of a $2.5dB$ and $3.7dB$ increase of the MSE of $\hat{G}$ and $\hat{Q}$ respectively.

6. CONCLUSION

Inexpensive and fast procedures for testing of instruments call for accurate estimation of parameters of a signal from relatively small number of samples. In this paper, it was shown that estimating parameters of dual channel sine-wave in which the signals from each channel are subsampled by half of paired coprime factors can reduce the demand on faster test procedures of RF electronic devices. Analysis of the estimation procedure and the CRB for the undersampled sequence is presented comparing with the results obtained from the seven parameter sine-fitting algorithm introduced in earlier works. Estimation of the common frequency required solving a set of four equations that depend on the frequency estimates from the subsampled channels.

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