

## SOME THOUGHTS ABOUT THE FREQUENTIST AND THE BAYESIAN APPROACH TO UNCERTAINTY

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**Abstract:** The paper presents ideas about the use of the frequentist and the Bayesian approach to estimation and uncertainty. The merits and the pitfalls of the Bayesian approach, compared with the frequentist one, are illustrated using a simple example (of electrical measurement), which gives rise to an instructive paradox. The implicit presence of the same paradox in the GUM is highlighted and discussed. The final suggestion is to keep in the GUM the approaches, the frequentist and the Bayesian one.

**Key words:** uncertainty, estimation theory, noise cancellation, Bayesian estimation, Stein paradox.

### 1 INTRODUCTION

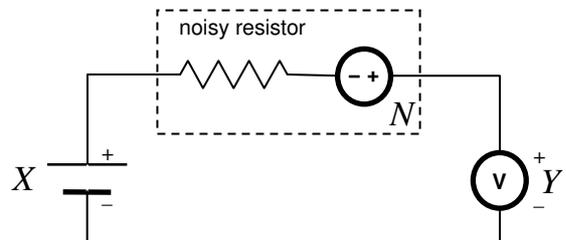
The "Guide to the Expression of Uncertainty in Measurement" (GUM) is currently made of separate pieces. The two main ones are the core document (GUM-100) [1], which dates back to 1993, and the Supplement 1 (GUM-101) [2], which came out in 2008. There are some differences in the conceptual framework of the two documents. GUM-100 does not prescribe explicitly a Bayesian approach to measurement uncertainty, even if the definition of uncertainty ("dispersion of the values that could reasonably be attributed to the measurand") seems to suggest a Bayesian view of the matter. According to some statisticians, GUM is actually a frequentist-Bayesian hybrid [3]. The latest edition of the International Vocabulary of Metrology (VIM) [4] is slightly more specific in taking the Bayesian viewpoint, for example when it is clearly stated that a "coverage interval" is not a confidence interval. The decisive and clearer document on the matter is, however, GUM-101. This document is very clear-cut in prescribing Bayesian definitions and procedures for measurement uncertainty; and ruling out frequentist concepts, e.g. that of unknown deterministic quantity. The selection of the appropriate uncertainty framework in metrology is, however, a debated issue [5], like the problem of validating independently an uncertainty assessment procedure [6].

The paper discusses the alternative between the frequentist approach (FA) and the Bayesian approach (BA) to measurement uncertainty. To this purpose, a simple example of frequentist and Bayesian measurement, leading to an instructive paradox, is described. Although the paradox, in a more general form, is already known and discussed in the literature [7], [8], the provided analysis is

original and relevant to the impact of the paradox on a GUM user.

### 2 A SIMPLE EXAMPLE OF FREQUENTIST AND BAYESIAN MEASUREMENT

Fig. 1 depicts a simple and idealized, yet realistic, measurement situation. The voltmeter yields the measurement  $Y = X + N$ , where  $X$  is the voltage of a battery and  $N$  is additive thermal noise. The thermal noise  $N$  is a normal random variable with zero mean and known variance  $\sigma^2$  (the assumption of known variance is not unrealistic as it can be computed from the resistor value, the absolute temperature, and the equivalent noise bandwidth of the system). The voltage  $X$  is known to be in the interval  $[a; b]$ . The interval is "large" with respect to the noise perturbing the measurement, i.e.  $b - a \gg \sigma$ , and no other prior information is available about  $X$ . The measured voltage is  $Y = y_0$ .



**Fig. 1. Measurement of a voltage with additive Gaussian noise.**

To fix ideas, consider the following numerical values:  $\sigma = 1$  mV;  $X \in [-10; +10]$  mV;  $y_0 = 2.5$  mV (units of measurement are inessential and will be dropped in the following).

In this Section, the differences between the frequentist and the Bayesian approach are briefly recalled, by assuming that the problem is estimating  $X$  from the knowledge of  $Y$ .

#### 2.1 Frequentist estimation and uncertainty of $X$

In the FA, the measurand  $X$  has an unknown fixed value  $x_0$ , while the measurement  $y_0$  is an outcome of the Gaussian random variable  $Y = x_0 + N$  (or, equivalently, of  $Y | X = x_0$  with  $Y = X + N$ ). The situation is depicted in Fig. 2, for the hypothetical value  $x_0 = 2$ .  $Y$  is normal with

mean  $x_0$  and therefore, with frequentist terminology, is an unbiased estimator of the fixed unknown number  $x_0$ . The goodness of the estimator  $Y$  is defined by the tightness of its distribution around the measurand  $x_0$ . In this framework, therefore, the only consistent definition of standard uncertainty is  $u = \text{std}(Y) = \sigma = 1$ . The expanded uncertainty is  $U = 2$  for a coverage factor  $k = 2$ . Since the distribution is normal, the associated probability is  $p \cong 95\%$ , and  $y_0 \pm U = 2.5 \pm 2$  is a (symmetric) confidence interval with a 95% confidence level.

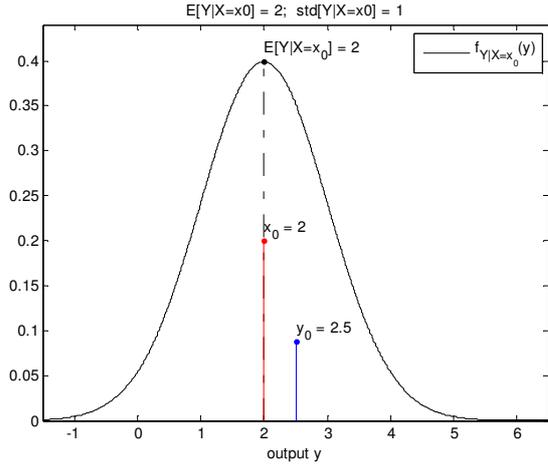


Fig. 2. Illustration of the estimation and uncertainty of  $X$  with the FA.

These well-known concepts are completely unacceptable according to GUM-101 and even to VIM. (It may be questioned if they are acceptable in the conceptual framework of GUM-100). In particular, it makes no sense to assert that the interval  $y_0 \pm U$  contains the value  $x_0$  "with a stated probability", as required by GUM-101 (definition 3.12) and VIM (definition 2.36), since all the involved quantities are deterministic (either the interval contains the value, or not).

In the FA the correct assertion is different and, of course, typically frequentist: if the measurement is repeated many times in identical conditions, the interval  $y_0 \pm u$  encompasses the fixed value  $x_0$  of the measurand in 95% of the cases, and it misses  $x_0$  in 5% of the cases. An equally valid assertion, which does not involve the concept of repeating measurements, is that *before making the measurement* the random interval  $Y \pm U$  had a 95% probability of encompassing  $x_0$ . The essence of frequentist estimation is not in repeating measurements, but in thinking the measurand as deterministic, and the measurement as a random variable.

## 2.2 Bayesian estimation and uncertainty of $X$

In the BA to uncertainty, the prior information about the measurand imposes to  $X$  a prior distribution,  $f_X(x)$ , uniform in  $[a; b]$ . The shape of the distribution is imposed by Jaynes' maximum entropy principle (MaxEnt), prescribed by GUM-101; in this very common case, MaxEnt is

equivalent to a simple and intuitive "indifference principle" (no value of  $X$  is considered more probable than others before making the measurement). If no measurement is done and no supplementary information is acquired, the best estimate of  $X$ , in the minimum mean squared error (MSE) sense, is  $E[X] = (b - a) / 2$ , and the standard uncertainty is  $u = \text{std}[X] = (b - a) / \sqrt{3}$ .

The distribution of  $X$  is updated by the measurement  $y_0$ , following the Bayes' rule:

$$f_{X|Y=y_0}(x) = \frac{l(x) \cdot f_X(x)}{\int_{-\infty}^{+\infty} l(x) \cdot f_X(x) dx} \quad (1)$$

where the likelihood function  $l(x) = f_{Y|X=x}(y_0)$  is, in this example, a Gaussian density centred in  $y_0 = 2.5$  with standard deviation  $\sigma = 1$ . The prior distribution  $f_X(x)$  is constant in an interval  $([a; b] = [-10; +10])$  that largely encompasses all the meaningful values of  $l(x)$  (roughly the interval  $[y_0 - 3\sigma; y_0 + 3\sigma] = [-0.5; 5.5]$ ). Therefore, with very good approximation,  $f_X(x)$  goes outside the integral (the typical case of a "non-informative prior"), yielding the result

$$f_{X|Y=y_0}(x) = l(x) = f_{Y|X=x}(y_0). \quad (2)$$

The posterior distribution (2) of  $X$  is reported in Fig. 3 (where also the "true value"  $x_0$  of the FA is represented). The best estimate of  $X$  in the minimum MSE sense is the expectation  $E[X | Y = y_0] = y_0 = 2.5$ , while the standard uncertainty is  $u = \text{std}[X | Y = y_0] = \sigma = 1$ . In this case the standard uncertainty  $u$  is actually the square root of the MSE of the estimate.

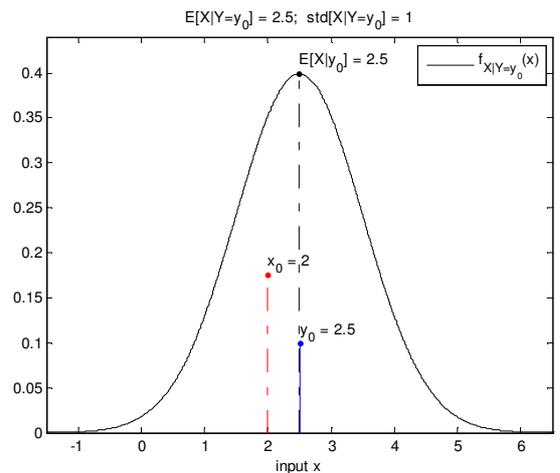


Fig. 3. Illustration of the estimation and uncertainty of  $X$  with the BA.

Like in the FA, the expanded uncertainty  $U = 2$  (coverage factor  $k = 2$ ) is associated with a probability  $p \cong 95\%$ . The big conceptual difference with respect to the

FA is that in this case it is correct to assert that  $P(X \in [y_0 - U; y_0 + U]) = p$ , or that the interval  $y_0 \pm U$  contains the measurand  $X$  with a stated probability. The interval is, with the VIM terminology, a coverage interval with a stated coverage probability; GUM-101 reports also the alternative (and more traditional) statistic terms "credible interval" and "Bayesian interval".

### 2.3 Considerations

In the example, the FA and the BA yields the same interval estimate,  $y_0 \pm U$ , with the same probability,  $p$ . Therefore, in this case, choosing one or the other approach is not a practical but a conceptual question. The BA has, however, three important features, which the example highlights:

- taking measurements is seen as a process of updating the information available about the measurand  $X$  ;
- it is not needed to introduce a deterministic value  $x_0$  of the measurand, a concept substituted by a sequence of narrower and narrower distributions of the measurand  $X$  (supposing to make more and more useful measurements);
- even when no measurement is available, an estimate and an uncertainty are attributed consistently to the measurand  $X$  on the basis of available information.

## 3 THE PARADOX ARISING FROM THE EXAMPLE

The difference between FA and BA ceases to be merely conceptual, and becomes impressively practical, if the measurement problem is slightly modified. In the same situation of Fig. 1, consider the problem of estimating both  $X$  and  $W = X^2$ , e.g. because the power dissipated in the resistor is also of interest. It must be noted that this is a very simple case of propagation, or of "mathematical model of measurement", according to the wording of GUM-100 and GUM-101.

### 3.1 Frequentist estimation and uncertainty of $W$

With the FA one must think the measurand as a fixed unknown value  $w_0 = x_0^2$ . It is natural to use  $Z = Y^2$  as an estimator of this quantity. However, it is readily seen that

$$E[Y^2] = E[(x_0 + N)^2] = x_0^2 + \sigma^2 \quad (3)$$

and therefore  $Y^2$  is a biased estimator of  $x_0^2$ . Since the variance is known, it is possible to obtain immediately a correct estimator

$$Z' = Y^2 - \sigma^2 \quad (4)$$

The variable  $Z = Y^2$  has a scaled non-central chi-square distribution, with 1 degree of freedom and noncentrality parameter  $\delta = w_0 / \sigma^2$ :

$$Z = \sigma^2 \chi_{nc}^2(1, \delta), \quad (5)$$

while  $Z'$  is of course  $Z$  shifted by  $\sigma^2$ . The distribution of  $Z'$  is depicted in Fig. 4, for the already considered hypothetical value  $x_0 = 2$ .

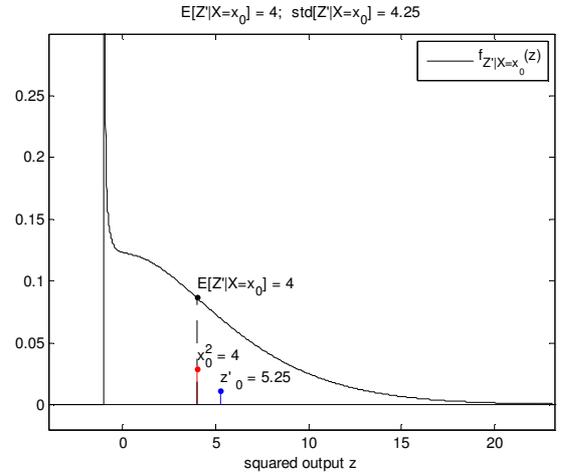


Fig. 4. Illustration of the measurement of  $X^2$  with the FA.

The unbiased estimator  $Z'$  (which in the example assumes the value  $z'_0 = 5.25$ ) has standard deviation  $\sigma_z = \sqrt{2\sigma^4 + 4x_0^2\sigma^2} \approx 4.24$ , the "frequentist" standard uncertainty. A confidence interval can be constructed using the quantiles of the noncentral chi-square distribution. In some cases (not in the example of Fig. 4), an approximate procedure involving the use of a Gaussian approximation is applicable, so that only the parameter  $\sigma_z$  is necessary. For the purpose of the analysis, however, it is important to highlight that the best estimate of  $w_0 = x_0^2$  is not  $y_0^2$ , but

$$\hat{w}_0 = y_0^2 - \sigma^2 \quad (6)$$

### 3.2 Bayesian estimation and uncertainty of $W$

The best Bayesian estimate in the minimum MSE sense can be readily calculated from the mean and the variance of the posterior distribution  $f_{X|Y=y_0}(x)$ :

$$E[X^2 | Y] = E[X | Y]^2 + \text{var}[X | Y]. \quad (7)$$

Therefore, with the BA, the estimate is:

$$\hat{w}_0 = y_0^2 + \sigma^2 \quad (8)$$

Also with the BA the estimate  $y_0^2$  is not correct, but the sign of the correction is *opposite* to that found with the FA. The posterior distribution of  $W = X^2$  requires the numerical computation of (1), and is reported in Fig. 5. The value of the estimate is  $E[X^2 | Y] = 7.25$ , and the value of the standard uncertainty is  $u = \text{std}[X^2 | Y = y_0] \approx 5.19$ . It must be noted that not only the estimate, but also the uncertainty is meaningfully different from the "frequentist" one.

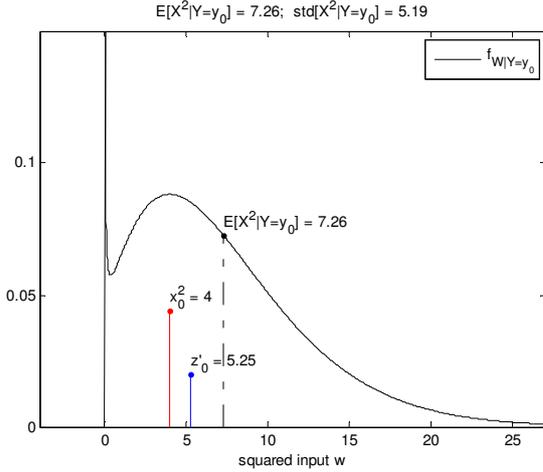


Fig. 5. Illustration of the measurement of  $X^2$  with the BA.

#### 4 COMMENTS ON THE PARADOX

While there is only a formal difference between FA and BA when estimating  $X$ , the difference is very substantial when estimating  $X^2$ . The paradox lies in the fact that the two estimates are both “optimal”, and they are not only different, but in a sense “opposite” (corrections of opposite sign,  $-\sigma^2$  vs.  $+\sigma^2$ ).

More specifically, if the measurement situation of Fig. 1 is simulated with a fixed input voltage  $x_0$ , the MSE of the frequentist estimator  $y_0^2 - \sigma^2$  is lower than that of the Bayesian estimator  $y_0^2 + \sigma^2$ , regardless the value of  $x_0$  (the first MSE is  $\sigma_z^2$ , the second  $\sigma_z^2 + 4\sigma^4$ ). This is not a surprise since in this “frequentist simulation” the two estimators have the same variance but the first one is unbiased. The superiority of the frequentist estimator for any given  $x_0$  seems to contradict the correctness of the Bayes formula, which assures minimum MSE for a uniformly distributed input. This is of course nonsense.

The paradox is, actually, a special case of a more general one, pointed out first (in a quite different form) by Stein [7], and discussed, for example, in [8]. The solution given in literature implies the withdrawal of MaxEnt, and the development of a new (non-uniform) prior distribution for  $X$  according to the more sophisticated machinery of Bernardo’s reference priors.

This solution is of great interest as an introduction to the theoretical body of objective Bayesianism, but does not fit the conceptual framework of the GUM. In GUM-101, the prior distribution is not influenced by the measurement model, but only by the available prior information, (according to rules given in clause 6); the measurement model must be used, instead, to evaluate the posterior distribution, using calculus or by means of Monte Carlo simulations (guidance given in clause 7).

Sticking by the GUM rules does not mean that the paradox cannot be solved. The following discussion aims at explaining the paradox in the GUM framework, and at highlighting the theoretical and practical consequences.

#### 4.1 Explanation of the paradox

Without having to change the prior distribution of  $X$ , the paradox has – of course – a straightforward solution, even if it can be not so immediate to figure out.

To appreciate the merits of the Bayesian estimator one must simulate the situation of Fig. 1 for a set of voltages  $x_0$  uniformly distributed in  $[a;b] = [-10;+10]$ . A superficial analysis suggests that this cannot change the evaluation, because of the “superiority for any fixed  $x_0$ ” of the frequentist estimator. It is not actually the case, because formula (8) applies only to outputs  $y_0$  well inside the interval  $[a;b]$ , and not near the borders. When  $y_0$  is near the borders or outside the interval, the actual Bayesian estimator departs from the simple formula  $y_0^2 + \sigma^2$  (Fig. 6). The difference is negligible in the interval  $y_0 \in [a + 2\sigma; b - 2\sigma]$ , but taking into account the changes near the borders makes a big final difference.

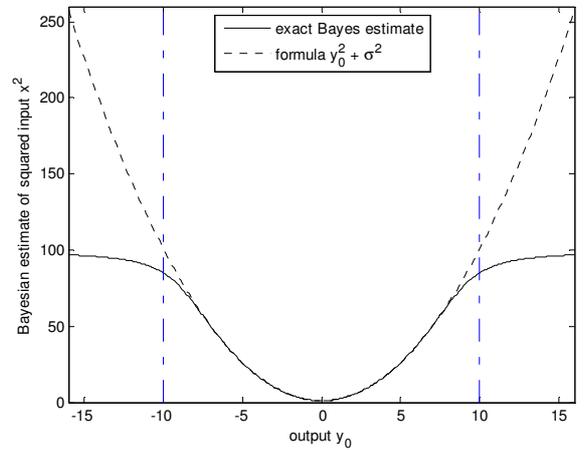


Fig. 6. Exact Bayesian estimation of  $X^2$  as a function of the output. The limits of the uniform distribution of the input are highlighted.

When the correct Bayes estimation is applied, the MSE of the frequentist estimator is no more better for any  $x_0 \in [a;b]$ , but only in a subinterval, as is clearly shown in Fig. 7. The frequentist estimator is better than the Bayesian one for (approximately)  $x_0 \in [a + 4\sigma; b - 4\sigma] = [-6;6]$ , i.e. for values of the input well inside the interval  $[a;b]$  (Fig. 8). In particular, the difference  $\Delta\text{MSE}(x_0)$  is equal to the predicted value  $-4\sigma^4 = -4$  in the interval  $[a + 5\sigma; b - 5\sigma] = [-5;5]$ . For other values of the input the Bayesian estimator is much superior, so that its overall MSE is below the MSE of the frequentist one, like it is theoretically expected to be (the difference being  $\Delta\text{MSE} \cong 34.7$ ).

Summing up, an analysis of the estimators for values of the input and/or the output near the borders of the interval explains the (apparent) paradox.

#### 4.2 Choosing between frequentist and Bayesian approach

The provided explanation does not settle the question of choosing between the two estimators in a practical

measurement situation like the one of the example, and like those addressed in the GUM. The Bayesian estimator is theoretically superior, but the following considerations against it come to attention. All of them are related to the fact that the Bayesian estimator is superior near the borders of the pdf of  $X$ , and inferior in the other cases (Fig. 7).

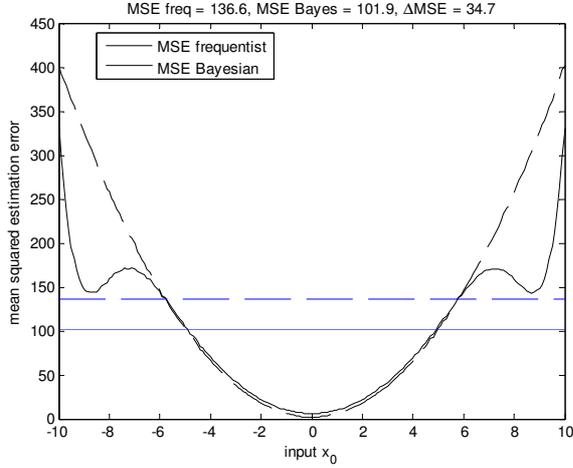


Fig. 7. MSE of the frequentist and the Bayesian estimators as a function of the input. The overall MSEs of the estimators (blue horizontal lines) are also represented.

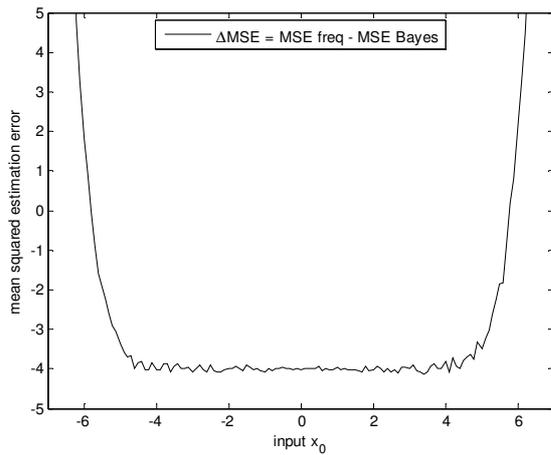


Fig. 8. Difference between the MSEs of the estimators for the input in the interval  $[-7;7]$

- 1) The Bayesian formula (8), in itself, is inferior to formula (6). If (8) is used *always* (independently on the output  $y_0$ ), the Bayesian MSE is higher. The BA is superior only when the exact Bayes estimate of Fig. 6 is used.
- 2) Therefore, fixing an interval  $[a,b]$  large enough to neglect border effects, so that (8) can be always used, is *inherently* inferior. (In this case one should change the prior distribution of  $X$  to obtain an admissible estimate).
- 3) The modified estimate near the borders, necessary for the superiority of the Bayes estimator, critically depends on the shape of the input pdf, in particular on the sharpness of the edges and on their location.

- 4) The input  $X$  has been given a uniform distribution on the basis of the mere prior information that  $X$  is in the interval  $[a;b]$ . Repeated measurements will prove the inferiority of the Bayesian estimator if  $X$  will not follow, actually, a uniform distribution with those exact edges.

The impression that, with “weak” prior information, the GUM Bayesian estimator is unjustified, is strengthened by the fact that the frequentist estimator is equivalent to a customary operation in electrical engineering. The power of a signal in additive noise is estimated by subtracting (and certainly not adding) the power of the noise from the power of the observed signal. Adding should be justified only when one knows exactly the limits between which lies, with uniform probability, the signal power.

Summing up:

- The GUM Bayes estimator  $y_0 + \sigma^2$  should be chosen only when the prior distribution of  $X$  is *actually known* to be uniform between the *exact* limits  $[a;b]$ , and when the different Bayes estimator computed for  $y_0$  near the borders is expected to be used in practice.
- The frequentist estimator  $y_0 - \sigma^2$  can be used in almost any other situation, in particular when the prior information is weaker, i.e. it is known only that  $X \in [a;b]$ .
- On objective Bayesian estimator, constructed with the appropriate prior, of course can always be used. But this is clearly outside the scope of the GUM.

Therefore, in a simple measurement problem addressed according to GUM rules, knowing that  $X$  should be in the interval  $[a;b]$ , without further information, is very different from knowing that  $X$  is uniformly distributed in  $[a;b]$ .

### 4.3 The paradox in the GUM

Annex H of GUM-100 reports an equation (numbered H.10) to compute an estimate of a nonlinear function of one or more quantities, taking into account the effect of variances and covariances in the nonlinearity. The equation is reported here literally:

$$y = f(\bar{X}_1, \bar{X}_2, \dots, \bar{X}_N) + \frac{1}{2} \sum_{i=1}^N \sum_{j=1}^N \frac{\partial^2 f}{\partial \bar{X}_i \partial \bar{X}_j} u(\bar{X}_i, \bar{X}_j) + \dots \quad (9)$$

(the averages are present because the formula is given in a context where the arguments are actually averages of five measured values).

In the example of this paper the nonlinear function is  $y = f(x) = x^2$ . With the GUM formula (9), the second-order term is equal to  $+\sigma^2$ . Direct application of the formula leads, therefore, to the Bayesian estimator (8). This is coherent with the fact that the formula is a propagation of random variables.

The same formula leads to a correction equal to  $-\sigma^2$ , and therefore to the frequentist estimator (6), if a frequentist

reasoning is followed. Consider the arguments  $\bar{X}_i$  to be unbiased frequentist estimators, equal to the sum of the measurand (true value)  $x_i$  and a zero-mean random variable  $E_i$  (the error). The output  $y = f(x_i + E_i)$  is an estimator whose expected value is given by (9), by substituting  $\bar{X}_i$  with  $x_i$ . In order to obtain an unbiased estimator, therefore, one must subtract the higher-order terms in (9):

$$y' = f(\bar{X}_1, \bar{X}_2, \dots, \bar{X}_N) - \frac{1}{2} \sum_{i=1}^N \sum_{j=1}^N \frac{\partial^2 f}{\partial \bar{X}_i \partial \bar{X}_j} u(\bar{X}_i, \bar{X}_j) - \dots \quad (10)$$

Therefore, GUM-100 itself contains the paradox, unfortunately without a clear indication whether one should choose the estimator  $\hat{w}_0 = y_0^2 + \sigma^2$ , or  $\hat{w}_0 = y_0^2 - \sigma^2$ . Interpreting (9) in a Bayesian way, the estimator  $\hat{w}_0 = y_0^2 + \sigma^2$  should be chosen. Unfortunately, the strong hypothesis of prior uniform distribution is not cited at all in Annex H of GUM-100, nor, of course, there are computations concerning the estimator near the borders of the distributions. Therefore, equation H.10 of GUM-100 is either frequentist, or Bayesian but incomplete, and hence of doubtful value.

## 5 SHOULD THE GUM CHOOSE BETWEEN FREQUENTISM AND BAYESIANISM?

The Bayesian-frequentist controversy spans in an enormous body of books, papers, web pages, with a vast number of examples, demonstrations, counter-examples and refutations. It is definitely out of place to discuss here the question in its general terms.

It is possible, instead, to reflect here on Bayesianism and frequentism in the GUM, i.e., in the definition of measurement uncertainty. The example illustrated in the paper can be more useful than other famous and more complex paradoxes, because of its simplicity and practical impact, because it arises in a very common measurement problem, and because it is in the GUM itself.

The first consideration coming from the example is that Bayesian estimation may be a delicate technical problem even in simple cases. If the prior distribution describes actual information on the input, which could be validated in a frequentist sense, the BA is of straightforward application; if, instead, there is none or poor prior information, the BA requires a particularly clever and technical construction of the right non-informative prior [8]. A mechanical application of the rules provided in GUM-101 leads to insidious pitfalls.

A second consideration is that one is not forced to choose once and for all between BA and FA. After all, uncountable scientific studies use frequentist estimation. According to authoritative specialists, it is probable that future challenging problems of statistical inference will be met by scientists with combinations of the two approaches [9]. It does not seem reasonable to rule out frequentism or Bayesianism; instead, it seems advisable to keep both the approaches, trying to give rules to use Bayesian estimation without “risks”.

## 6 CONCLUSIONS

The paper presents some ideas about the two rival approaches to estimation and uncertainty evaluation: the frequentist and the Bayesian approach. The ideas are presented by means of an example, which illustrates quite clearly both the merits and the pitfalls of the Bayesian approach. An apparent paradox, arising from the example and leading to “opposite” ways of estimating the measurand, is pointed out and explained. The presence of the paradox in the GUM, and consequently of a serious ambiguity on its frequentist or Bayesian nature, is pointed out.

The ideas presented can be summarized by saying that ruling out frequentism and imposing Bayesianism, like in the Supplement 1 to GUM, does not seem a wise choice. The reason is that when poor prior information is available, Bayesian methods require much more theory than that available in Supplement 1 (and also considerable personal skill as a statistician). When, on the contrary, actual (frequentist) prior information on the measurand is available, Bayesian methods do not present pitfalls, and are instead quite simple, direct and powerful.

## REFERENCES

- [1] International Organization for Standardization, *ISO/IEC Guide 98-3:2008 - Uncertainty of measurement – Part 3: Guide to the expression of uncertainty in measurement*. ISO, 2008.
- [2] —, *ISO/IEC Guide 98-3:2008/Suppl 1:2008 - Uncertainty of measurement – Propagation of distributions using a Monte Carlo method*. ISO, 2008.
- [3] B. Toman and W. Guthrie, “Assessing uncertainty in measurement,” in *proc. of “A two-day workshop on Bayesian methods that frequentists should know”*, College Park, MD, Apr. 2008.
- [4] International Organization for Standardization, *ISO/IEC Guide 99:2007 - International vocabulary of metrology – Basic and general concepts and associated terms (VIM)*. ISO, 2007.
- [5] J. Alves Sousa and A. Silva Ribeiro, “The choice of method to the evaluation of measurement uncertainty in metrology,” in *Proc. of XIX IMEKO World Congress*, Lisbon, Portugal, Sep. 6–11, 2009.
- [6] B. D. Hall, “Evaluating methods of calculating measurement uncertainty,” *Metrologia*, vol. 45, pp. L5–L8, Mar. 2008.
- [7] C. Stein, “An example of wide discrepancy between fiducial and confidence intervals,” *Annals of Mathematical Statistics*, vol. 30, no. 4, pp. 877–880, 1959.
- [8] A. R. Syversveen, “Noninformative bayesian priors. interpretation and problems with construction and applications.” 1998. [Online]. Available: <http://www.math.ntnu.no/preprint/statistics/1998/S3-1998.ps>
- [9] B. Efron. (2005) Modern science and the Bayesian-frequentist controversy. [Online]. Available: [www-stat.stanford.edu/brad/papers/NEW-ModSci2005.pdf](http://www-stat.stanford.edu/brad/papers/NEW-ModSci2005.pdf)